Binary Images and Their Foreground Polyhedra
Thinning Algorithms
Thinning: The Topology Preservation Requirement
Homology-Simple Sets
In 2D & 3D Cartesian Grids, Homology-Simple 1 = (8,4)/-(26,6)- Simple 1
Seq-Homology-Simple & Hereditarily Homology-Simple Sets
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Acyclic Polyhedra
Cores of $\mathcal{F}$-\$s; Homology-Critical $\mathcal{F}$-\$s
$\mathcal{P}$-Homology-Simple Elements
Main Theorem 1: Characterization of $\mathcal{P}$-Homology-Simpleness
Attachment Sets
Main Theorem 2: Characterization of Hereditary Homology-Simpleness

Main Theorem 1 ⇒ Main Theorem 2

Another Statement of Main Theorems 1 & 2

Strongly Normal (SN) Collections

Restatement of Main Thms. 1 & 2 in Terms of Cliques When $\mathcal{F}$ is SN

Proof of Main Thm. 1: 2 ⇒ 1

Proof of Main Thm. 1: 1 ⇒ 2

Summary

Definition of $\text{Attach}(P, \mathcal{L})$
Definition of $\text{Core}_{\mathcal{F}}(C)$
Definition of $\mathcal{F}_C$
Definition of $\text{hereditarily homology-simple}$
Definition of $\text{F-homology-critical}$
Definition of $\text{homology-cosimple and hereditarily homology-cosimple}$
Definition of $\text{homology-simple}$
Definition of $\mathcal{F} \cap (\mathcal{F}\text{-intersection})$
Definition of $\text{seq-homology-simple}$
Hereditarily Homology-Simple Sets and Homology Critical Kernels of Binary Images on Sets of Convex Polytopes

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What is This Talk About (1)?

A **convex polytope** is a set that is the convex hull of a finite set of points in some Euclidean space $\mathbb{R}^n$.

A **polyhedron** is a set that is the union of a finite collection of convex polytopes in a Euclidean space.

- The union of any finite collection of polyhedra is a polyhedron.
- The intersection of any finite collection of polyhedra is a polyhedron.

- This talk will present **homology-critical** kernels, which are a variant of Bertrand's **critical** kernels: When dealing with sets of grid cells of a 2D, 3D, or 4D Cartesian grid, **homology-critical** and **critical** are equivalent.
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- This talk will present \textit{homology-critical} kernels, which are a variant of Bertrand's \textit{critical} kernels: When dealing with sets of grid cells of a 2D, 3D, or 4D Cartesian grid, \textit{homology-critical} and \textit{critical} are equivalent.

- Many results about \textit{critical} kernels of such sets become valid for sets of \textit{arbitrary convex polytopes of any dimension} (and, more generally, sets of \textit{arbitrary acyclic polyhedra whose nonempty intersections are acyclic}) if they are restated as results about \textit{homology-critical} kernels.
• Below is an (extremely simple) 2D example of a set of polyhedra to which the main results of this talk would apply.

• While these five 2D polyhedra have disjoint interiors, our main results are also valid for collections of polyhedra whose interiors overlap.

From:
G. T. Herman,

*Figure 4.1.1. A simple digital space.*
Another 2D example of a set of polyhedra to which the main results of this talk would apply:

- The polyhedra here are the 2D convex polytopes bounded by the gray lines.
- The green parts of this drawing are irrelevant.
What is This Talk About (2)?

A **thinning algorithm** simplifies a binary image by reducing its foreground to a thin "skeleton" in a "topology-preserving" way. One formulation of the "topology-preserving" requirement is that the set of deleted image elements satisfy the condition of being **homology-simple** (a term we will define) in the image foreground $\mathcal{F}$. 
What is This Talk About (2)?

A *thinning algorithm* simplifies a binary image by reducing its foreground to a thin "skeleton" in a "topology-preserving" way. One formulation of the "topology-preserving" requirement is that the set of deleted image elements satisfy the condition of being *homology-simple* (a term we will define) in the image foreground $\mathcal{F}$.

A methodology due to Bertrand and Couprie, based on critical kernels, designs parallel thinning algorithms that always satisfy this requirement.
What is This Talk About (2)?

A *thinning algorithm* simplifies a binary image by reducing its foreground to a thin "skeleton" in a "topology-preserving" way. One formulation of the "topology-preserving" requirement is that the set of deleted image elements satisfy the condition of being *homology-simple* (a term we will define) in the image foreground $\mathcal{F}$.

A methodology due to Bertrand and Couprie, based on critical kernels, designs parallel thinning algorithms that always satisfy this requirement. For binary images on *grid cells of a 2D, 3D, or 4D Cartesian grid*, a fundamental theorem of Bertrand and Couprie relating to critical kernels provides a useful local necessary and sufficient condition for every subset of a given set of image elements to be homology-simple in $\mathcal{F}$.

• Our main result (Main Theorem 2) substitutes *homology-critical* for *critical* in a statement of this theorem and so gives an analogous necessary and sufficient condition that is valid for binary images on sets of arbitrary convex polytopes of any dimension—even if some of the polytopes have overlapping interiors—and, still more generally, sets of arbitrary acyclic polyhedra whose nonempty intersections are acyclic.
Binary Images and Their Foreground Polyhedra

Let $\mathcal{G}$ be a set of polyhedra—e.g., $\mathcal{G}$ may be a set of grid cells of a $n$D Cartesian grid—and let $I : \mathcal{G} \rightarrow \{0, 1\}$ be such that $I^{-1}[\{1\}]$ is finite. Then:
Binary Images and Their Foreground Polyhedra

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- The function $I$ will be called a **binary image** on $\mathcal{G}$.
- If $P \in \mathcal{G}$ and $I(P) = 1$, then we say $P$ is a **1** of $I$.
- If $P \in \mathcal{G}$ and $I(P) = 0$, then we say $P$ is a **0** of $I$. 

Binary Images and Their Foreground Polyhedra

Let \( \mathcal{G} \) be a set of polyhedra—e.g., \( \mathcal{G} \) may be a set of grid cells of a \( n \)D Cartesian grid—and let \( I: \mathcal{G} \rightarrow \{0, 1\} \) be such that \( I^{-1}[\{1\}] \) is finite. Then:

- The function \( I \) will be called a **binary image** on \( \mathcal{G} \).
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- If \( P \in \mathcal{G} \) and \( I(P) = 0 \), then we say \( P \) is a **0** of \( I \).
- The **foreground** of \( I \) is the set \( I^{-1}[\{1\}] \) — i.e., the set of all \( 1 \)s of \( I \). This set will be denoted by \( \mathcal{F}_I \).
- The **foreground polyhedron** of \( I \) is the set \( \bigcup \mathcal{F}_I = \bigcup I^{-1}[\{1\}] \).

Left: A binary image \( I \) on a set of 88 grid cells of a 2D Cartesian grid.

Right: \( I \)'s foreground polyhedron \( \mathcal{F}_I \)
Thinning Algorithms

A *thinning algorithm* is used to transform a binary image by reducing its foreground to a thin "skeleton".

Let $I^{\text{in}} : G \rightarrow \{0, 1\}$ and $I^{\text{out}} : G \rightarrow \{0, 1\}$ be the input and output binary images of an nD thinning algorithm.

Thinning algorithms change 1s to 0s but never change 0s to 1s, so the foreground of $I^{\text{out}}$ is a subset of the foreground of $I^{\text{in}}$: $\mathcal{F}_{I^{\text{out}}} \subseteq \mathcal{F}_{I^{\text{in}}}$

3 Examples of 3D Thinning (Using Different Thinning Algorithms)
Topological Requirements of Thinning: Homology-Simpleness

We expect 2D thinning algorithms to preserve connected components and internal cavities of $\mathcal{F}_{\text{in}}$.

We expect 3D thinning algorithms to preserve connected components, internal cavities, and holes/tunnels of $\mathcal{F}_{\text{in}}$.

The following condition $T$ states these requirements precisely, and also generalizes them to higher-dimensional thinning:

$T$: The inclusion $\iota : \bigcup \mathcal{F}_{\text{out}} \rightarrow \bigcup \mathcal{F}_{\text{in}}$ must be a homology isomorphism—$
\iota$ must induce a group isomorphism $\iota_* : H_k(\bigcup \mathcal{F}_{\text{out}}) \rightarrow H_k(\bigcup \mathcal{F}_{\text{in}})$ for all $k$.

Let $\mathcal{F}$ be any set of polyhedra, let $\mathcal{D} \subseteq \mathcal{F}$ and let $\mathcal{Q} \in \mathcal{F}$. Then we say $\mathcal{D}$ is homology-simple in $\mathcal{F}$ if the inclusion $\bigcup (\mathcal{F} \setminus \mathcal{D}) \rightarrow \bigcup \mathcal{F}$ is a homology isomorphism. We say $\mathcal{Q}$ is homology-simple in $\mathcal{F}$ if $\{\mathcal{Q}\}$ is.
Topological Requirements of Thinning: Homology-Simpleness

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Let $\mathcal{F}$ be any set of polyhedra, let $\mathcal{D} \subseteq \mathcal{F}$ and let $Q \in \mathcal{F}$. Then we say $\mathcal{D}$ is **homology-simple** in $\mathcal{F}$ if the inclusion $\bigcup (\mathcal{F} \setminus \mathcal{D}) \to \bigcup \mathcal{F}$ is a homology isomorphism. We say $Q$ is **homology-simple** in $\mathcal{F}$ if $\{Q\}$ is.

So the topological condition $T$ can also be stated as follows:

**T**: The set $\mathcal{F}_{in} \setminus \mathcal{F}_{out}$ must be homology-simple in $\mathcal{F}_{in}$.
Homology-Simple Sets in the Plane

If $\mathcal{F}$ is any finite set of polyhedra in the plane $\mathbb{R}^2$ and $\mathcal{D} \subseteq \mathcal{F}$, then $\mathcal{D}$ is homology-simple in $\mathcal{F}$ if and only if none of the following occurs when we remove $\mathcal{D}$ from $\mathcal{F}$:

1. A component of $\bigcup \mathcal{F}$ is split.
   [e.g., $\mathcal{D} = \{E,F\}$ is not homology-simple in $\mathcal{F}$.]

2. A component of $\bigcup \mathcal{F}$ is eliminated.
   [e.g., $\mathcal{D} = \{G,H,I,J\}$ is not homology-simple in $\mathcal{F}$.]

3. A component of $\bigcup \mathcal{F}$ gains a new internal cavity.
   [e.g., $\mathcal{D} = \{C\}$ is not homology-simple in $\mathcal{F}$.]

4. A component of $\bigcup \mathcal{F}$ loses an internal cavity when that internal cavity is merged with another cavity or merged with the component's outside. [e.g., $\{A\}$ and $\{B, C, D\}$ are not homology-simple in $\mathcal{F}$.]
Homology-Simple Sets in $\mathbb{R}^n$

More generally, if $\mathcal{F}$ is a set of polyhedra in $\mathbb{R}^n$ and $\mathcal{D} \subseteq \mathcal{F}$, then $\mathcal{D}$ is homology-simple in $\mathcal{F}$ just if there is no $k \leq n$ such that removal of $\mathcal{D}$ from $\mathcal{F}$ splits or eliminates a class of homologous $k$-dimensional cycles.

[Two $k$-cycles $z$ and $z'$ in a set $X$ are said to be homologous (in $X$) just if there exists a $(k+1)$-chain $c$ in $X$ such that the boundary of $c$ is $z - z'$.]

$\mathcal{D}$ is homology-simple in $\mathcal{F}$ just if neither of the following is true:

1. For some $k \leq n$, $\exists$ non-homologous $k$-cycles of $\bigcup(\mathcal{F} \setminus \mathcal{D})$ that are homologous in $\bigcup \mathcal{F}$.

Deletion of $\mathcal{D}$ splits the class of 1-cycles of $\bigcup \mathcal{F}$ that are homologous to the blue and the green 1-cycles: This “creates a hole.”
Homology-Simple Sets in $\mathbb{R}^n$

More generally, if $F$ is a set of polyhedra in $\mathbb{R}^n$ and $D \subseteq F$, then $D$ is homology-simple in $F$ just if there is no $k \leq n$ such that removal of $D$ from $F$ splits or eliminates a class of homologous $k$-dimensional cycles.

[Two $k$-cycles $z$ and $z'$ in a set $X$ are said to be homologous (in $X$) just if there exists a $(k+1)$-chain $c$ in $X$ such that the boundary of $c$ is $z - z'$.]

$D$ is homology-simple in $F$ just if neither of the following is true:

1. For some $k \leq n$, $\exists$ non-homologous $k$-cycles of $\bigcup(F \setminus D)$ that are homologous in $\bigcup F$.

2. For some $k \leq n$, $\exists$ a $k$-cycle in $\bigcup F$ that is not homologous to any $k$-cycle in $\bigcup (F \setminus D)$.

The mapping of homology classes of $k$-cycles induced by the inclusion $\iota : \bigcup(F \setminus D) \rightarrow \bigcup F$ is not 1-1 in case 1, not onto in case 2.
Recall Let $\mathcal{F}$ be any finite set of polyhedra, let $\mathcal{D} \subseteq \mathcal{F}$ and let $Q \in \mathcal{F}$. Then we say $\mathcal{D}$ is **homology-simple** in $\mathcal{F}$ if the inclusion $\bigcup(\mathcal{F} \setminus \mathcal{D}) \to \bigcup \mathcal{F}$ is a homology isomorphism. We say $Q$ is **homology-simple** in $\mathcal{F}$ if $\{Q\}$ is.

**Homology-Simpleness in 2D and 3D Cartesian Grids**

Most applications of binary images use binary images $\Gamma : \mathcal{G} \to \{0, 1\}$ for which $\mathcal{G}$ is a set of grid cells of a 2D or 3D Cartesian grid (so that $\Gamma$'s foreground $\mathcal{F}_\Gamma = \Gamma^{-1}[\{1\}]$ is a set of grid cells of the same Cartesian grid).

When $\mathcal{F}$ is a set of grid cells of a 2D or 3D Cartesian grid and $Q \in \mathcal{F}$, it can be shown that **the following are equivalent:**

1. $Q$ is homology-simple in $\mathcal{F}$.

2. $Q$ is a simple element of $\mathcal{F}$ in the "traditional" (8,4) or (26,6) sense.

Regarding 2, various **local** characterizations of elements $Q$ that are simple in traditional senses have been given by a number of authors — e.g., Rosenfeld (1970) in the 2D case, and Morgenthaler (1981), Tsao+Fu (1982), Saha et al. / Bertrand+Malandain (1991/2), Kong (1995), Bertrand (1996), and Bertrand+Couprie (2006) in the 3D case.
Recall Let \( \mathcal{F} \) be any finite set of polyhedra, let \( \mathcal{D} \subseteq \mathcal{F} \) and let \( Q \in \mathcal{F} \). Then we say \( \mathcal{D} \) is **homology-simple** in \( \mathcal{F} \) if the inclusion \( \bigcup (\mathcal{F} \setminus \mathcal{D}) \to \bigcup \mathcal{F} \) is a homology isomorphism. We say \( Q \) is **homology-simple** in \( \mathcal{F} \) if \( \{Q\} \) is.

**Seq-Homology-Simpleness & Hereditary Homology-Simpleness**

Let \( \mathcal{F} \) be any finite set of polyhedra, and let \( \mathcal{D} \subseteq \mathcal{F} \).

We say \( \mathcal{D} \) is **sequentially-homology-simple** or **seq-homology-simple** in \( \mathcal{F} \) if there is an enumeration \( Q_1, ..., Q_k \) of the elements of \( \mathcal{D} \) such that:

- \( Q_i \) is homology-simple in \( \mathcal{F} \setminus \{Q_1, ..., Q_{i-1}\} \) for \( 1 \leq i \leq k \).
Recall Let $\mathcal{F}$ be any finite set of polyhedra, let $\mathcal{D} \subseteq \mathcal{F}$ and let $Q \in \mathcal{F}$. Then we say $\mathcal{D}$ is **homology-simple** in $\mathcal{F}$ if the inclusion $\bigcup (\mathcal{F} \setminus \mathcal{D}) \to \bigcup \mathcal{F}$ is a homology isomorphism. We say $Q$ is **homology-simple** in $\mathcal{F}$ if $\{Q\}$ is.

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$\mathcal{D}$ is seq-homology-simple in $\mathcal{F}$ $\Rightarrow$ $\mathcal{D}$ is homology-simple in $\mathcal{F}$.

as homology isomorphisms are closed under composition.
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But, even in $\mathbb{R}^3$,

$\mathcal{D}$ is homology-simple in $\mathcal{F}$ $\not\Rightarrow$ $\mathcal{D}$ is seq-homology-simple in $\mathcal{F}$.

If $|\mathcal{F}| > 1$ and $\bigcup \mathcal{F}$ is acyclic but no element of $\mathcal{F}$ is homology-simple in $\mathcal{F}$ (as is possible even if $\mathcal{F}$ is a set of cubical voxels) then, for any acyclic $Q \in \mathcal{F}$, $\mathcal{F} \setminus \{Q\}$ is homology-simple but not seq-homology-simple in $\mathcal{F}$.
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We say $\mathcal{D}$ is \textit{sequentially-homology-simple} or \textit{seq-homology-simple} in $\mathcal{F}$
if there is an enumeration $Q_1, \ldots, Q_k$ of the elements of $\mathcal{D}$ such that:

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$\mathcal{D}$ is seq-homology-simple in $\mathcal{F}$ $\implies$ $\mathcal{D}$ is homology-simple in $\mathcal{F}$.

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$\mathcal{D}$ is seq-homology-simple in $\mathcal{F} \implies \mathcal{D}$ is homology-simple in $\mathcal{F}$.

as homology isomorphisms are closed under composition.

But, even in $\mathbb{R}^3$, $\mathcal{D}$ is homology-simple in $\mathcal{F}$ $\nRightarrow$ $\mathcal{D}$ is seq-homology-simple in $\mathcal{F}$.

If $|\mathcal{F}| > 1$ and $\bigcup \mathcal{F}$ is acyclic but no element of $\mathcal{F}$ is homology-simple in $\mathcal{F}$ (as is possible even if $\mathcal{F}$ is a set of cubical voxels) then, for any acyclic $Q \in \mathcal{F}$, $\mathcal{F} \setminus \{Q\}$ is homology-simple but not seq-homology-simple in $\mathcal{F}$.

We say $\mathcal{D}$ is \textit{hereditarily homology-simple} in $\mathcal{F}$ if

every subset of $\mathcal{D}$ is homology-simple in $\mathcal{F}$.

We say $\mathcal{D}$ is \textit{hereditarily seq-homology-simple} in $\mathcal{F}$ if

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We will see that:

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Seq-Homology-Simplessness & Hereditary Homology-Simpleness

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We say $\mathcal{D}$ is **sequentially-homology-simple** or **seq-homology-simple** in $\mathcal{F}$ if there is an enumeration $Q_1, ..., Q_k$ of the elements of $\mathcal{D}$ such that:

- $Q_i$ is homology-simple in $\mathcal{F} \setminus \{Q_1, ..., Q_{i-1}\}$ for $1 \leq i \leq k$.

We say $\mathcal{D}$ is **hereditarily homology-simple** in $\mathcal{F}$ if every subset of $\mathcal{D}$ is homology-simple in $\mathcal{F}$.
We say $\mathcal{D}$ is **hereditarily seq-homology-simple** in $\mathcal{F}$ if every subset of $\mathcal{D}$ is seq-homology-simple in $\mathcal{F}$.

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We say $\mathcal{D}$ is **sequentially-homology-simple** or **seq-homology-simple** in $\mathcal{F}$
if there is an enumeration $Q_1, \ldots, Q_k$ of the elements of $\mathcal{D}$ such that:
- $Q_i$ is homology-simple in $\mathcal{F} \setminus \{Q_1, \ldots, Q_{i-1}\}$ for $1 \leq i \leq k$.

We say $\mathcal{D}$ is **hereditarily homology-simple** in $\mathcal{F}$ if
*every* subset of $\mathcal{D}$ is homology-simple in $\mathcal{F}$.
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We will see that: $\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$
\[\iff\] $\mathcal{D}$ is hereditarily seq-homology-simple in $\mathcal{F}$
\[\iff\] for *every* enumeration $Q_1, \ldots, Q_k$ of the elements of $\mathcal{D}$
$Q_i$ is homology-simple in $\mathcal{F} \setminus \{Q_1, \ldots, Q_{i-1}\}$ for $1 \leq i \leq k$

When $\mathcal{F}$ is a set of grid cells of a 2D or 3D Cartesian grid and $Q \in \mathcal{F}$, since
$Q$ is homology-simple in $\mathcal{F} \iff Q$ is simple in $\mathcal{F}$ in the "traditional" sense
"$\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$" can be understood **purely in**
**terms of simpleness in the traditional** (8,4) or (26,6) **sense**!
Digression: (4,8) or (6,26)-Simple 1s and Homology-Cosimple Sets

Let $I : G \to \{0, 1\}$ be a binary image on a collection $G$ of polyhedra, let $D \subseteq F_I = I^{-1}[\{1\}]$ and let $Q \in F_I$. Thus $G \setminus F_I = I^{-1}[\{0\}]$.

Then we say $D$ is **homology-cosimple** in $F_I$ if $D$ is homology-simple in $(G \setminus F_I) \cup D$. We say $Q$ is **homology-cosimple** in $F_I$ if $\{Q\}$ is.

We say $D$ is **hereditarily** homology-cosimple in $F_I$ if every subset of $D$ is.

- When $G$ is the set of all grid cells of a 2D or 3D Cartesian grid and $Q \in F_I$, it can be shown that **the following are equivalent**:
  1. $Q$ is homology-cosimple in $F_I$.
  2. $Q$ is a simple element of $F_I$ in the traditional (4,8) or (6,26) sense.
Digression: (4,8) or (6,26)-Simple 1s and Homology-Cosimple Sets

Let $I : G \rightarrow \{0, 1\}$ be a binary image on a collection $G$ of polyhedra, let $\mathcal{D} \subseteq \mathcal{F}_I = I^{-1}[\{1\}]$ and let $Q \in \mathcal{F}_I$. Thus $G \setminus \mathcal{F}_I = I^{-1}[\{0\}]$.

Then we say $\mathcal{D}$ is **homology-cosimple** in $\mathcal{F}_I$ if $\mathcal{D}$ is homology-simple in $(G \setminus \mathcal{F}_I) \cup \mathcal{D}$. We say $Q$ is **homology-cosimple** in $\mathcal{F}_I$ if $\{Q\}$ is.

We say $\mathcal{D}$ is **hereditarily** homology-cosimple in $\mathcal{F}_I$ if every subset of $\mathcal{D}$ is.

- When $G$ is the set of all grid cells of a 2D or 3D Cartesian grid and $Q \in \mathcal{F}_I$, it can be shown that **the following are equivalent**:
  1. $Q$ is homology-cosimple in $\mathcal{F}_I$.
  2. $Q$ is a simple element of $\mathcal{F}_I$ in the traditional (4,8) or (6,26) sense.

- When $G$ is a locally finite collection of convex polytopes (or, more generally, acyclic polyhedra whose nonempty intersections are acyclic), the local characterization (in terms of homology-critical kernels) of hereditarily homology-*simple* sets $\mathcal{D}$ given by our Main Thm. 2 implies a local characterization of hereditarily homology-*cosimple* sets $\mathcal{D}$, since it can be shown that: $\mathcal{D}$ is hereditarily homology-cosimple in $\mathcal{F}_I$ if and only if $\mathcal{D}$ is hereditarily homology-simple in $(G \setminus \mathcal{F}_I) \cup \mathcal{D}$.
Thinning algorithms must satisfy the following topological requirement: 
**T**: The set $\mathcal{F}_{\text{in}} \setminus \mathcal{F}_{\text{out}}$ must be homology-simple in $\mathcal{F}_{\text{in}}$.

**Pseudocode of a Typical Parallel Thinning Algorithm**

1. $I = I_{\text{in}}$
2. while the termination condition is not satisfied do
3. $D = \text{a subset of } \mathcal{F}_I \text{ that is hereditarily homology-simple in } \mathcal{F}_I$
4. $I = I - D$
5. $I_{\text{out}} = I$
Recall

Thinning algorithms must satisfy the following topological requirement:

\[ T: \text{The set } F_{\text{in}} \setminus F_{\text{out}} \text{ must be homology-simple in } F_{\text{in}}. \]

**Pseudocode of a Typical Parallel Thinning Algorithm**

1. \( I = I^{\text{in}} \)
2. **while** the termination condition is *not* satisfied **do**
3. \( D = \) a subset of \( F_I \) that is *hereditarily homology-simple* in \( F_I \)
4. \( I = I - D \)
5. \( I^{\text{out}} = I \)

- In step 4, \( I - D \ \overset{\text{def}}{=} \text{the binary image on } G \text{ whose set of } 1s \text{ is } F_I \setminus D. \)
- **T** is satisfied, as \( D \) is homology-simple in \( F_I \) at every loop iteration.
Recall: Thinning algorithms must satisfy the following topological requirement:

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- The subsets \( D \) are chosen to satisfy some **non**-topological requirements:
  - *The shape of* \( I^{\text{out}} \)'s **foreground** should reflect that of \( I^{\text{in}} \)'s **foreground**.
  - \( I^{\text{out}} \)'s **foreground** should be well centered relative to \( I^{\text{in}} \)'s **foreground**.
  - \( I^{\text{out}} \)'s **foreground** should be very thin.

Such requirements are very important, but are not the focus of this talk.
Pseudocode of a Typical Parallel Thinning Algorithm

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\[ \text{For the topology preservation condition} \ T \textit{to be satisfied, the set} \ D \textit{that is deleted at each iteration need only be homology-simple.} \]

\[ \text{But algorithms in which} \ D \textit{is} \textit{hereditarily} \textit{homology-simple at all iterations are more common.} \]
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- For the topology preservation condition $T$ to be satisfied, the set $D$ that is deleted at each iteration need only be homology-simple.
- But algorithms in which $D$ is hereditarily homology-simple at all iterations are more common.
- For images on Cartesian grids, methodologies that verify or ensure $D$ is always hereditarily homology-simple have been developed since the '70s by, e.g., Rosenfeld, Ronse, Hall (2D), Bertrand, Ma, Bertrand & Couprie.
Critical Kernels and $\mathcal{F}$-\text{n}s ($\mathcal{F}$-Intersections)

Critical kernels, introduced by Bertrand (2005)—and extensively used and studied by Bertrand and Couprie—provide a powerful methodology for developing parallel thinning algorithms each of whose iterations is guaranteed to delete a hereditarily homology-simple set.

- Suppose for example that $K_I$ is a subset of the foreground $F_I$ at some iteration and (to satisfy non-topological requirements) we wish to preserve $K_I$.
- We can use the critical kernel of $F_I$ to find a relatively large set $D$ of elements of $F_I \setminus K_I$ that is hereditarily homology-simple in $F_I$, so that deletion of $D$ preserves $K_I$ and satisfies the topology-preservation condition.

Let $\mathcal{F}$ be any finite collection of nonempty sets. An $\mathcal{F}$-intersection or $\mathcal{F}$-\text{n} is a nonempty set $S$ such that $S = \bigcap C$ for some nonempty subcollection $C$ of $\mathcal{F}$. Here $C$ may consist of 1 member of $\mathcal{F}$: Each member of $\mathcal{F}$ is an $\mathcal{F}$-\text{n}.

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Critical Kernels and $\mathcal{F}$-$\cap$s ($\mathcal{F}$-Intersections)

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Let $\mathcal{F}$ be any finite collection of nonempty sets. An $\mathcal{F}$-intersection or $\mathcal{F}$-$\cap$ is a nonempty set $S$ such that $S = \bigcap C$ for some nonempty subcollection $C$ of $\mathcal{F}$. Here $C$ may consist of 1 member of $\mathcal{F}$: Each member of $\mathcal{F}$ is an $\mathcal{F}$-$\cap$.

Note: If each of $A$ and $B$ is an $\mathcal{F}$-$\cap$ and $A \cap B \neq \emptyset$, then $A \cap B$ is an $\mathcal{F}$-$\cap$. 
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The critical kernel of a set $\mathcal{F}$ of grid cells of a Cartesian grid is determined by a set of $\mathcal{F}$-$\cap$s called **critical** $\mathcal{F}$-$\cap$s:

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- When $\mathcal{F}$ is a finite set of grid cells of a 2D, 3D, or 4D Cartesian grid, a theorem of Bertrand & Couprie (2009) characterizes a minimal non-simple subset of $\mathcal{F}$ as a subset of $\mathcal{F}$ that is the “clique” induced by an inclusion-maximal critical $\mathcal{F}$-$\cap$.
- If $\mathcal{F}$ is a finite set of grid cells of a 2D, 3D, or 4D Cartesian grid and $D \subseteq \mathcal{F}$, then

$$D \text{ is hereditarily homology-simple in } \mathcal{F} \iff D \text{ contains no minimal non-simple subset of } \mathcal{F}$$

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\[ \text{THEOREM (Bertrand & Couprie)} \quad \text{If } \mathcal{F} \text{ is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid, and } \mathcal{D} \subseteq \mathcal{F}, \text{ then the following are equivalent:} \]

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Conversely, when $\mathcal{F}$ is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid one can deduce Bertrand and Couprie's characterization of the minimal non-simple subsets of $\mathcal{F}$ from this theorem (and the fact that a subset $\mathcal{T}$ of $\mathcal{F}$ is a minimal non-simple subset of $\mathcal{F}$ if and only if (i) $\mathcal{T}$ is not hereditarily homology-simple in $\mathcal{F}$, but (ii) every proper subset of $\mathcal{T}$ is hereditarily homology-simple in $\mathcal{F}$).

If $\mathcal{T}$ is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid, then Bertrand & Couprie's characterization of minimal non-simple sets implies
Recall: From Bertrand & Couprie's characterization of the minimal non-simple subsets of a finite set of grid cells of a 2D, 3D, or 4D Cartesian grid we can deduce:

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If $\mathcal{T}$ is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid, then Bertrand & Couprie's characterization of minimal non-simple sets implies

$$\exists \mathcal{F}. \; \mathcal{T} \text{ is a minimal non-simple set of } \mathcal{F} \iff \mathcal{F} \cap \mathcal{T} \neq \emptyset$$

and (using another theorem from Bertrand & Couprie (2009)) also implies

$$\exists \mathcal{F}. \; (\mathcal{T} \text{ is a minimal non-simple set of } \mathcal{F} \cap \mathcal{T} \text{ consists of more and } \bigcup \mathcal{T} \text{ is not a component of } \bigcup \mathcal{F}) \iff \mathcal{T} \text{ consists of more than just one point.}$$
Recall: From Bertrand & Couprie's characterization of the minimal non-simple subsets of a finite set of grid cells of a 2D, 3D, or 4D Cartesian grid we can deduce:

**Theorem (Bertrand & Couprie)** If $F$ is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid, and $D \subseteq F$, then the following are equivalent:

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If $T$ is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid, then Bertrand & Couprie's characterization of minimal non-simple sets implies

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$\exists F. (T$ is a minimal non-simple set of $F \iff \cap T$ consists of more and $\cup T$ is not a component of $\cup F) \iff$ than just one point.

- These facts were originally proved over many years by Ronse (1988, 2D), Ma (1994, 3D), Kong (1995, 3D), and Gau & Kong (2003, 4D).
Recall: **Theorem** (Bertrand & Couprie) If $\mathcal{F}$ is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid, and $\mathcal{D} \subseteq \mathcal{F}$, then the following are equivalent:

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**Generalizing the Bertrand-Couprie Theorem**

Bertrand & Couprie's $\mathcal{F}$-critical $\mathcal{F} \cap$s are defined in terms of *collapsing* of subcomplexes of the cubical complex whose set of facets is $\mathcal{F}$. *But we are going to use a more general concept: $\mathcal{F}$-homology-critical $\mathcal{F} \cap$s.*
Recall: **Theorem** (Bertrand & Couprie) *If* $\mathcal{F}$ *is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid, and* $\mathcal{D} \subseteq \mathcal{F}$, *then the following are equivalent:*

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Recall: **Theorem (Bertrand & Couprie)** If $\mathcal{F}$ is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid, and $\mathcal{D} \subseteq \mathcal{F}$, then the following are equivalent:

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Unlike $\mathcal{F}$-critical $\mathcal{F} \cap \mathcal{D}$s, $\mathcal{F}$-homology-critical $\mathcal{F} \cap \mathcal{D}$s are defined in a way that *doesn't* depend on the existence of a complex whose set of facets is $\mathcal{F}$: The definition is valid even if the interiors of some members of $\mathcal{F}$ overlap.

It follows from results of Couprie & Bertrand (2009) and Kong (1997) that if $\mathcal{F}$ is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid then:

$\mathcal{F}$-homology-critical $\iff \mathcal{F}$-critical

**Hence:**
Recall: **Theorem** (Bertrand & Couprie) *If \( F \) is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid, and \( D \subseteq F \), then the following are equivalent:

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\[ F \text{-homology-critical } \Leftrightarrow F \text{-critical} \]

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Generalizing the Bertrand-Couprie Theorem

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Bertrand & Couprie's $\mathcal{F}$-critical $\mathcal{F}$-\n are defined in terms of \textit{collapsing} of subcomplexes of the cubical complex whose set of facets is $\mathcal{F}$. \textit{But we are going to use a more general concept: $\mathcal{F}$-homology-critical $\mathcal{F}$-\n}s. Unlike $\mathcal{F}$-critical $\mathcal{F}$-\ns, $\mathcal{F}$-homology-critical $\mathcal{F}$-\ns are defined in a way that \textit{doesn't} depend on the existence of a complex whose set of facets is $\mathcal{F}$: The definition is valid even if the interiors of some members of $\mathcal{F}$ overlap.

It follows from results of Couprie & Bertrand (2009) and Kong (1997) that if $\mathcal{F}$ is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid then:

\[ \mathcal{F}\text{-homology-critical} \Leftrightarrow \mathcal{F}\text{-critical} \]

\textbf{Hence: Theorem (Bertrand & Couprie)} \textit{If $\mathcal{F}$ is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid, and $\mathcal{D} \subseteq \mathcal{F}$, then the following are equivalent:}

1. $\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$.
2. Every $\mathcal{F}$-homology-critical $\mathcal{F}$-\n is contained in a member of $\mathcal{F} \setminus \mathcal{D}$.

\textbf{• Our main result is that this} version of the theorem is valid much more generally: It is valid when $\mathcal{F}$ is \textit{any finite set of acyclic polyhedra whose nonempty intersections are acyclic.} For example, it is valid when $\mathcal{F}$ is \textit{any finite set of convex polytopes.}
Acyclic Polyhedra

A *convex polytope* is a set that is the convex hull of a finite set of points in some Euclidean space $\mathbb{R}^n$.

A *polyhedron* is a set that is the union of a finite collection of convex polytopes in a Euclidean space.

- The union of any finite collection of polyhedra is a polyhedron.
- The intersection of any finite collection of polyhedra is a polyhedron.

A set $P$ is said to be *acyclic* if

1. $P$ is nonempty and connected, and
2. $P$ has trivial homology in all positive dimensions (intuitively, "$P$ has no holes of any dimension").
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In the plane $\mathbb{R}^2$, a polyhedron $P$ is acyclic if and only if

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In $\mathbb{R}^3$, a polyhedron $P$ is acyclic if and only if the following are all true:
  1. $P$ is nonempty and connected.
  2. $\mathbb{R}^3 \setminus P$ is connected—i.e., $P$ has no internal cavities.
  3. The Euler characteristic of $P$ is 1.

When 1 and 2 hold, 3 holds if and only if $P$ "has no holes or tunnels" (and if and only if $P$ is simply connected).
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When 1 and 2 hold, 3 holds if and only if $P " has no holes or tunnels"$ (and if and only if $P$ is simply connected).

In $\mathbb{R}^n$ (for any dimension $n$), we have that:
• Any convex polytope is acyclic; more generally, if $C$ is any nonempty collection of convex sets such that $\bigcap C \neq \emptyset$, then $\bigcup C$ is acyclic.
• If $P$ and $Q$ are two acyclic polyhedra such that $P \cap Q$ is acyclic, then $P \cup Q$ is also acyclic.

We say a set $G$ of polyhedra is *good* if $G$ is finite, each member of $G$ is acyclic, and every nonempty intersection of $\geq 2$ members of $G$ is acyclic.
**Example:** *Any finite set of convex polytopes is a good set of polyhedra.*
\(F\)-Cores of \(F\)-\(\cap\)s; \(F\)-Homology-Critical \(F\)-\(\cap\)s

Let \(\mathcal{F}\) be any finite collection of nonempty sets. Recall that an \(\mathcal{F}\)-\(\cap\) is a nonempty set \(S\) such that \(S = \bigcap C\) for some nonempty subcollection \(C\) of \(\mathcal{F}\). [\(C\) may consist of just one member of \(\mathcal{F}\): Any member of \(\mathcal{F}\) is an \(\mathcal{F}\)-\(\cap\)!]

We now define the \(\mathcal{F}\)-core of an \(\mathcal{F}\)-\(\cap\).

This concept is very similar to Bertrand's concept of the core of a cell of a complex (but does not refer to any complex).
\textbf{\textit{\(\mathcal{F}\)-Cores of \(\mathcal{F}\)-\(\cap\)s; \(\mathcal{F}\)-Homology-Critical \(\mathcal{F}\)-\(\cap\)s}}

Let \(\mathcal{F}\) be any finite collection of nonempty sets. Recall that an \(\mathcal{F}\)-\(\cap\) is a \textit{nonempty} set \(S\) such that \(S = \cap C\) for some nonempty subcollection \(C\) of \(\mathcal{F}\). [\(C\) may consist of just one member of \(\mathcal{F}\): Any member of \(\mathcal{F}\) is an \(\mathcal{F}\)-\(\cap\)!]

We now define the \textbf{\(\mathcal{F}\)-core} of an \(\mathcal{F}\)-\(\cap\).

This concept is very similar to Bertrand's concept of the \textit{core} of a cell of a complex (but does not refer to any complex).

If \(C\) is an \(\mathcal{F}\)-\(\cap\), then we define: \(\text{Core}_{\mathcal{F}}(C) \triangleq C \cap \bigcup \{F \in \mathcal{F} \mid F \not\supset C\}\)

Thus: \(\text{Core}_{\mathcal{F}}(C) = \) the intersection of \(C\) with the union of those members of \(\mathcal{F}\) that do not contain \(C\).

We call \(\text{Core}_{\mathcal{F}}(C)\) the \textbf{\(\mathcal{F}\)-core} of \(C\).
**F-Cores of F-∩s; F-Homology-Critical F-∩s**

Let \( \mathcal{F} \) be any finite collection of nonempty sets. Recall that an \( \mathcal{F} \)-∩ is a **nonempty** set \( S \) such that \( S = \bigcap \mathcal{C} \) for some nonempty subcollection \( \mathcal{C} \) of \( \mathcal{F} \). [\( \mathcal{C} \) may consist of just one member of \( \mathcal{F} \): Any member of \( \mathcal{F} \) is an \( \mathcal{F} \)-∩!]

We now define the **\( \mathcal{F} \)-core** of an \( \mathcal{F} \)-∩.

This concept is very similar to Bertrand's concept of the *core* of a cell of a complex (but does not refer to any complex).

If \( \mathcal{C} \) is an \( \mathcal{F} \)-∩, then we define: \( \text{Core}_\mathcal{F}(\mathcal{C}) \triangleq \mathcal{C} \cap \bigcup \{ \mathcal{F} \in \mathcal{F} | \mathcal{F} \not\subseteq \mathcal{C} \} \)

Thus: \( \text{Core}_\mathcal{F}(\mathcal{C}) = \text{the intersection of} \ \mathcal{C} \ \text{with the union of those members of} \ \mathcal{F} \ \text{that do not contain} \ \mathcal{C} \).

We call \( \text{Core}_\mathcal{F}(\mathcal{C}) \) the **\( \mathcal{F} \)-core** of \( \mathcal{C} \).

An \( \mathcal{F} \)-∩ \( \mathcal{C} \) is said to be **\( \mathcal{F} \)-homology-critical** if \( \text{Core}_\mathcal{F}(\mathcal{C}) \) is **not** acyclic.

**Hence:** An \( \mathcal{F} \)-∩ is \( \mathcal{F} \)-homology-critical if and only if its \( \mathcal{F} \)-core is ∅, or is disconnected, or has nontrivial homology in some positive dimension.
Three Examples of $\mathcal{F}$-\(\cap\)s That are \textit{NOT} \(\mathcal{F}\)-Homology-Critical

\(=\) element of \(\mathcal{F}\)

Consider the 3 \textbf{orange} \(\mathcal{F}\)-\(\cap\)s and their \(\mathcal{F}\)-cores (\textbf{colored blue}).

In each case, the \(\mathcal{F}\)-core is \textit{nonempty}, \textit{is connected}, \textit{has no hole}, and \textit{has no internal cavity}.

\(\therefore\) \textbf{none} of these 3 \(\mathcal{F}\)-\(\cap\)s is \(\mathcal{F}\)-homology-critical!
Six Examples of $\mathcal{F}$-Homology-Critical $\mathcal{F}$-$\cap$s

$= \text{element of } \mathcal{F}$

Consider the 6 orange $\mathcal{F}$-$\cap$s and their $\mathcal{F}$-cores (colored blue):

In each case, the $\mathcal{F}$-core

*either* is empty *or* is disconnected.

$\therefore$ each of these 6 $\mathcal{F}$-$\cap$s is $\mathcal{F}$-homology-critical!
If $\mathcal{F}$ is the set of five cubes that are are shown here, and $C$ is \textit{this} (transparent) cube

... then $\text{Core}_\mathcal{F}(C)$ is the \textbf{blue} set.

$\text{Core}_\mathcal{F}(C)$ \textit{has a hole}.

$\therefore C$ \textit{is $\mathcal{F}$-homology-critical}!

---

If $\mathcal{F}$ is the set of four cubes that are are shown here, and $C$ is \textit{this} (transparent) cube

... then $\text{Core}_\mathcal{F}(C)$ is the \textbf{blue} set.

$\text{Core}_\mathcal{F}(C)$ \textit{is nonempty}, \textit{is connected}, \textit{has no hole}, and \textit{has no internal cavity}.

$\therefore C$ \textit{is not $\mathcal{F}$-homology-critical}!
Recall: If \( C \) is any \( \mathcal{F} \)-\( \cap \), then

\[
\text{Core}_{\mathcal{F}}(C) \overset{\text{def}}{=} C \cap \bigcup \{ F \in \mathcal{F} \mid F \not\supseteq C \} \\
= \text{the intersection of } C \text{ with the union of those members of } \mathcal{F} \text{ that do } \textit{not} \text{ contain } C.
\]

An \( \mathcal{F} \)-\( \cap \) \( C \) is said to be \( \mathcal{F} \)-homology-critical if \( \text{Core}_{\mathcal{F}}(C) \) is \( \textit{not} \) acyclic.

\( \therefore \) An \( \mathcal{F} \)-\( \cap \) is \( \mathcal{F} \)-homology-critical \( \textit{if and only if} \) its \( \mathcal{F} \)-core is \( \emptyset \), or is disconnected, or has nontrivial homology in some positive dimension.

We define the \textit{homology-critical kernel} of \( \mathcal{F} \) to be the set of all \( \mathcal{F} \)-homology-critical \( \mathcal{F} \)-\( \cap \)s.

Notes: 1. If \( \mathcal{F} \) is a set of grid cells of a 2D, 3D, or 4D Cartesian grid, then an \( \mathcal{F} \)-\( \cap \) is \( \mathcal{F} \)-homology-critical \( \textit{if and only if} \) it is \( \mathcal{F} \)-critical in the sense of Bertrand and Couprie. (This follows from results established by Couprie & Bertrand (2009) and Kong (1997).)

2. If \( C \) is any \( \mathcal{F} \)-\( \cap \), then it is readily confirmed that:

\[
\text{Core}_{\mathcal{F}}(C) = \bigcup \{ C \cap F \mid F \in \mathcal{F} \text{ and } F \not\supseteq C \} \\
= \bigcup \{ Y \mid Y \text{ is an } \mathcal{F} \text{-}\( \cap \text{ and } Y \subseteq C \} \\
= \text{the union of the } \mathcal{F} \text{-}\( \cap \text{s strictly contained in } C \).
**P-Homology-Simple Elements**

Let $\mathcal{F}$ be a finite set of polyhedra. If $Q \in \mathcal{D} \subseteq \mathcal{F}$, then we say $Q$ is **P-homology-simple** for $\mathcal{D}$ in $\mathcal{F}$ if the following is true:

- $Q$ is homology-simple in $\mathcal{F} \setminus S$ for all $S \subseteq \mathcal{D} \setminus \{Q\}$.

This definition is a straightforward generalization of a concept that was originally defined by Bertrand (1995).

**Note:** If $Q \in \mathcal{D'} \subseteq \mathcal{D} \subseteq \mathcal{F}$ and $Q$ is P-homology-simple for $\mathcal{D}$ in $\mathcal{F}$, then it is evident that $Q$ is also P-homology-simple for $\mathcal{D'}$ in $\mathcal{F}$. 
**ℙ-Homology-Simple Elements**

Let \( \mathcal{F} \) be a finite set of polyhedra. If \( Q \in \mathcal{D} \subseteq \mathcal{F} \), then we say \( Q \) is **ℙ-homology-simple** for \( \mathcal{D} \) in \( \mathcal{F} \) if the following is true:

- \( Q \) is homology-simple in \( \mathcal{F} \setminus S \) for all \( S \subseteq \mathcal{D} \setminus \{Q\} \).

This definition is a straightforward generalization of a concept that was originally defined by Bertrand (1995).

**Note:** If \( Q \in \mathcal{D}' \subseteq \mathcal{D} \subseteq \mathcal{F} \) and \( Q \) is ℙ-homology-simple for \( \mathcal{D} \) in \( \mathcal{F} \), then it is evident that \( Q \) is also ℙ-homology-simple for \( \mathcal{D}' \) in \( \mathcal{F} \).

We will see that: **\( \mathcal{D} \) is hereditarily homology-simple in \( \mathcal{F} \) if and only if** every element of \( \mathcal{D} \) is ℙ-homology-simple for \( \mathcal{D} \) in \( \mathcal{F} \).

Now consider the subset of \( \mathcal{D} \) defined by:

\[
\mathbb{P}(\mathcal{D}, \mathcal{F}) = \{Q \in \mathcal{D} \mid Q \text{ is ℙ-homology-simple for } \mathcal{D} \text{ in } \mathcal{F}\}
\]

From the case \( \mathcal{D}' = \mathbb{P}(\mathcal{D}, \mathcal{F}) \) of the above Note, we see that **every element of \( \mathbb{P}(\mathcal{D}, \mathcal{F}) \) is ℙ-homology-simple for \( \mathbb{P}(\mathcal{D}, \mathcal{F}) \) in \( \mathcal{F} \).** Hence:

For any \( \mathcal{D} \subseteq \mathcal{F} \), the set \( \mathbb{P}(\mathcal{D}, \mathcal{F}) \) is hereditarily homology-simple in \( \mathcal{F} \).
A *Local* Characterization of $\mathbb{P}$-Homology-Simpleness

Our 1st main result characterizes $\mathbb{P}$-homology-simpleness *locally*, in terms of $\mathcal{F}$-homology-critical $\mathcal{F}$-$\cap$s.

When $\mathcal{F}$ is a set of grid cells of a 2D, 3D, or 4D Cartesian grid, one can deduce this theorem from a theorem of Bertrand & Couprie (2009), since one can show [using results of Couprie & Bertrand (2009)] that in this case "$\mathcal{F}$-homology-critical" and "$\mathbb{P}$-homology-simple" are equivalent to the concepts of "critical" and "$\mathbb{P}$-simple" used by Bertrand & Couprie:
A Local Characterization of $\mathbb{P}$-Homology-Simpleness

Our 1st main result characterizes $\mathbb{P}$-homology-simpleness \textit{locally}, in terms of $\mathcal{F}$-homology-critical $\mathcal{F}$-$\cap$s.

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**MAIN THEOREM 1** Let $\mathcal{F}$ be any finite set of polyhedra such that every $\mathcal{F}$-$\cap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:

1. $Q$ is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$.
2. Every $\mathcal{F}$-homology-critical $\mathcal{D}$-$\cap$ contained in $Q$ is also contained in a member of $\mathcal{F} \setminus \mathcal{D}$.

Note: Condition 2 $\iff$ Every $\mathcal{F}$-homology-critical $\mathcal{F}$-$\cap$ contained in $Q$ is also contained in a member of $\mathcal{F} \setminus \mathcal{D}$.

since any $\mathcal{F}$-$\cap$ that's not a $\mathcal{D}$-$\cap$ is evidently contained in a member of $\mathcal{F} \setminus \mathcal{D}$!
Attachment Set of a Polyhedron in a Set of Polyhedra

If $P$ is a polyhedron and $\mathcal{L}$ a set of polyhedra then we define

$$\text{Attach}(P, \mathcal{L}) \overset{\text{def}}{=} P \cap \bigcup (\mathcal{L} \setminus \{P\})$$

and we call this set the $\mathcal{L}$-attachment set of $P$. Note that:

1. $\text{Attach}(P, \mathcal{L}) = \text{Attach}(P, \mathcal{L} \cup \{P\}) = \text{Attach}(P, \mathcal{L} \setminus \{P\})$
2. If $P \notin \mathcal{L}$, then $\text{Attach}(P, \mathcal{L}) = P \cap \bigcup \mathcal{L}$.
3. If $P$ is an inclusion-maximal member of $\mathcal{L}$, $\text{Attach}(P, \mathcal{L}) = \text{Core}_\mathcal{L}(P)$.
4. If $P$ is not an inclusion-maximal member of $\mathcal{L}$, $\text{Attach}(P, \mathcal{L}) = P$. 
Attachment Set of a Polyhedron in a Set of Polyhedra

If $P$ is a polyhedron and $\mathcal{L}$ a set of polyhedra then we define

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4. If $P$ is not an inclusion-maximal member of $\mathcal{L}$, $\text{Attach}(P, \mathcal{L}) = P$.

If $\mathcal{L}$ is the set of pale gray and dark gray squares on the right, then the $\mathcal{L}$-attachment set or $\mathcal{L}$-core of any pale gray square is the union of its black 0- and 1-faces.

If $\mathcal{L}$ is the set of 6 cubes shown below, and $P$ is this cube

$$\ldots \text{ then } \text{Attach}(P, \mathcal{L}) = \text{Core}_\mathcal{L}(P)$$

is the union of the 0-, 1-, and 2-faces that are colored black here:
In the sequel, $\mathcal{F}$ denotes a finite set of acyclic polyhedra.  
(Many later results will further assume that every $\mathcal{F} \cap$ is acyclic.)

**Proposition 1:** Let $Q \in \mathcal{L} \subseteq \mathcal{F}$. Then the following are equivalent:

(a) $Q$ is homology-simple in $\mathcal{L}$.  
(b) $\text{Attach}(Q, \mathcal{L})$ is acyclic.

**Note:** If $Q$ is *inclusion-maximal* in $\mathcal{L}$, then $\text{Attach}(Q, \mathcal{L}) = \text{Core}_\mathcal{L}(Q)$ and so $Q$ is homology-simple in $\mathcal{L} \iff Q$ is *not* $\mathcal{L}$-homology-critical.
In the sequel, \( \mathcal{F} \) denotes a finite set of acyclic polyhedra.
(Many later results will further assume that every \( \mathcal{F} \cap \) is acyclic.)

Proposition 1: Let \( Q \in \mathcal{L} \subseteq \mathcal{F} \). Then the following are equivalent:
(a) \( Q \) is homology-simple in \( \mathcal{L} \).  
(b) \( \text{Attach}(Q, \mathcal{L}) \) is acyclic.

Note: If \( Q \) is inclusion-maximal in \( \mathcal{L} \), then \( \text{Attach}(Q, \mathcal{L}) = \text{Core}_{\mathcal{L}}(Q) \) and so \( Q \) is homology-simple in \( \mathcal{L} \) \iff \( Q \) is not \( \mathcal{L} \)-homology-critical.

Corollary 2: Let \( Q \in \mathcal{D} \subseteq \mathcal{F} \). Then \( Q \) is \( \mathbb{P} \)-homology-simple for \( \mathcal{D} \) in \( \mathcal{F} \) if and only if \( \text{Attach}(Q, \mathcal{F} \setminus S) \) is acyclic for all \( S \subseteq \mathcal{D} \).
In the sequel, \( \mathcal{F} \) denotes a finite set of acyclic polyhedra. (Many later results will further assume that every \( \mathcal{F} \)-\( \cap \) is acyclic.)

**Proposition 1:** Let \( Q \in \mathcal{L} \subseteq \mathcal{F} \). Then the following are equivalent:

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**Note:** If \( Q \) is inclusion-maximal in \( \mathcal{L} \), then \( \text{Attach}(Q, \mathcal{L}) = \text{Core}_\mathcal{L}(Q) \) and so \( Q \) is homology-simple in \( \mathcal{L} \) \( \iff \) \( Q \) is not \( \mathcal{L} \)-homology-critical.

**Corollary 2:** Let \( Q \in \mathcal{D} \subseteq \mathcal{F} \). Then \( Q \) is \( \mathbb{P} \)-homology-simple for \( \mathcal{D} \) in \( \mathcal{F} \) if and only if \( \text{Attach}(Q, \mathcal{F} \setminus S) \) is acyclic for all \( S \subseteq \mathcal{D} \).

Prop. 1 follows from the fact that \( \text{Attach}(Q, \mathcal{L}) = Q \cap \bigcup(\mathcal{L} \setminus \{Q\}) \) and results of topology: Reduced homology sequences and the Excision Thm.

\[
\begin{align*}
0 & \to \widetilde{H}_p(Q) \to \widetilde{H}_p(Q, \text{Attach}(Q, \mathcal{L})) \to \widetilde{H}_{p-1}((\text{Attach}(Q, \mathcal{L})) \to \widetilde{H}_{p-1}(Q) \to \cdots \\
\text{by Excision} & \\end{align*}
\]
**Lemma 3**: Let $\mathcal{T} \subseteq \mathcal{F}$ and let $Q \in \mathcal{F} \setminus \mathcal{T}$. Then all of the following are true if any two are true:

A. $\mathcal{T}$ is homology-simple in $\mathcal{F}$.

B. $\mathcal{T} \cup \{Q\}$ is homology-simple in $\mathcal{F}$.

C. $Q$ is homology-simple in $\mathcal{F} \setminus \mathcal{T}$.

- The conclusion of Lemma 3 remains true—with the same proof—if we replace $\{Q\}$ and $Q$ in B and C with *any subset* $\mathcal{T}'$ of $\mathcal{F} \setminus \mathcal{T}$! (But we only need the special case that is stated in the lemma.)

- From this lemma we can deduce the following previously stated fact:

  $\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$

  $\iff \mathcal{D}$ is hereditarily seq-homology-simple in $\mathcal{F}$

  $\iff \text{for every enumeration } Q_1, \ldots, Q_k \text{ of the elements of } \mathcal{D}$

  $Q_i$ is homology-simple in $\mathcal{F} \setminus \{Q_1, \ldots, Q_{i-1}\}$ for $1 \leq i \leq k$
RECALL: Lemma 3: Let $S \subseteq \mathcal{F}$ and let $Q \in \mathcal{F} \setminus S$. Then all of the following are true if any two are true:

A. $S$ is homology-simple in $\mathcal{F}$.
B. $S \cup \{Q\}$ is homology-simple in $\mathcal{F}$.
C. $\{Q\}$ is homology-simple in $\mathcal{F} \setminus S$.

Proposition 4: Let $\mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:

1. $\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$.
2. For all $Q \in \mathcal{D}$ and $S \subseteq \mathcal{D} \setminus \{Q\}$,
   
   $S \cup \{Q\}$ is homology-simple in $\mathcal{F}$ \textbf{if} $S$ is homology-simple in $\mathcal{F}$.
3. For all $Q \in \mathcal{D}$ and $S \subseteq \mathcal{D} \setminus \{Q\}$, $\{Q\}$ is homology-simple in $\mathcal{F} \setminus S$.
4. Every $Q \in \mathcal{D}$ is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$.

Proof: $2 \Rightarrow 1$ by induction, because $\emptyset$ is homology-simple in $\mathcal{F}$.

$3 \Rightarrow 2$ and $1 \Rightarrow 3$ both follow from Lemma 3.

$3 \Leftrightarrow 4$ follows from the definition of $\mathbb{P}$-homology-simple. //
RECALL: MAIN THEOREM 1 Let $\mathcal{F}$ be a finite set of polyhedra such that every $\mathcal{F} \cap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:

1. $Q$ is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$.
2. Every $\mathcal{F}$-homology-critical $\mathcal{D} \cap$ contained in $Q$ is also contained in a member of $\mathcal{F} \setminus \mathcal{D}$.

Proposition 4: Let $\mathcal{D} \subsetneq \mathcal{F}$. Then the following are equivalent:

1. $\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$.
4. Each $Q \in \mathcal{D}$ is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$.

A Local Characterization of Hereditarily Homology-Simple Sets

From Main Theorem 1 and the equivalence of statements 1 and 4 of Proposition 4 we deduce:

MAIN THEOREM 2: Let $\mathcal{F}$ be any finite set of acyclic polyhedra such that every $\mathcal{F} \cap$ is acyclic, and let $\mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:

1. $\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$.
2. Every $\mathcal{F}$-homology-critical $\mathcal{D} \cap$ is contained in a member of $\mathcal{F} \setminus \mathcal{D}$.
A Local Characterization of Hereditarily Homology-Simple Sets

From Main Theorem 1 and the equivalence of statements 1 and 4 of Proposition 4 we deduce:

**MAIN THEOREM 2:** Let $\mathcal{F}$ be any finite set of acyclic polyhedra such that every $\mathcal{F} \cap$ is acyclic, and let $\mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:

1. $\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$.
2. Every $\mathcal{F}$-homology-critical $\mathcal{D} \cap$ is contained in a member of $\mathcal{F} \setminus \mathcal{D}$. 

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A *Local* Characterization of Hereditarily Homology-Simple Sets

From Main Theorem 1 and the equivalence of statements 1 and 4 of Proposition 4 we deduce:

**MAIN THEOREM 2:** Let $\mathcal{F}$ be any finite set of acyclic polyhedra such that every $\mathcal{F}$-$\cap$ is acyclic, and let $\mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:

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$\equiv$

2 $\iff$ Every $\mathcal{F}$-homology-critical $\mathcal{F}$-$\cap$ is contained in a member of $\mathcal{F} \setminus \mathcal{D}$.

since an $\mathcal{F}$-$\cap$ that's not a $\mathcal{D}$-$\cap$ is evidently contained in a member of $\mathcal{F} \setminus \mathcal{D}$!

If no member of $\mathcal{F}$ contains another member of $\mathcal{F}$, then condition 2 implies that no member of $\mathcal{D}$ is $\mathcal{F}$-homology-critical

or, equivalently, that every member of $\mathcal{D}$ is homology-simple in $\mathcal{F}$
Recall: **MAIN THM. 1** Let $\mathcal{F}$ be any finite set of polyhedra such that every $\mathcal{F}$-$\cap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:

1. $Q$ is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$.
2. Every $\mathcal{F}$-homology-critical $\mathcal{D}$-$\cap$ contained in $Q$ is also contained in a member of $\mathcal{F} \setminus \mathcal{D}$.

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**Notation:** If $C$ is any $\mathcal{F}$-$\cap$, we define $\mathcal{F}_C \triangleq \{ F \in \mathcal{F} \mid F \supseteq C \}$ (so $\cap \mathcal{F}_C = C$).

We now restate Main Theorems 1 & 2 in terms of these sets $\mathcal{F}_C$: 
Recall: **MAIN THM. 1** Let $\mathcal{F}$ be any finite set of polyhedra such that every $\mathcal{F}$-$\cap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:
1. $Q$ is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$.
2. Every $\mathcal{F}$-homology-critical $\mathcal{D}$-$\cap$ contained in $Q$ is also contained in a member of $\mathcal{F} \setminus \mathcal{D}$.

**MAIN THM. 2:** Let $\mathcal{F}$ be any finite set of polyhedra such that every $\mathcal{F}$-$\cap$ is acyclic, and let $\mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:
1. $\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$.
2. Every $\mathcal{F}$-homology-critical $\mathcal{D}$-$\cap$ is contained in a member of $\mathcal{F} \setminus \mathcal{D}$.

**Notation:** If $C$ is any $\mathcal{F}$-$\cap$, we define $\mathcal{F}_C \overset{\text{def}}{=} \{F \in \mathcal{F} \mid F \supseteq C\}$ (so $\cap \mathcal{F}_C = C$).

We now restate Main Theorems 1 & 2 in terms of these sets $\mathcal{F}_C$:

**THEOREM:** Let $\mathcal{F}$ be any finite set of polyhedra such that every $\mathcal{F}$-$\cap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then:

(i) $Q$ is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$ **if and only if** there is no $\mathcal{F}$-homology-critical $\mathcal{F}$-$\cap$ $C$ such that $Q \in \mathcal{F}_C \subseteq \mathcal{D}$.

(ii) $\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$ **if and only if** there is no $\mathcal{F}$-homology-critical $\mathcal{F}$-$\cap$ $C$ such that $\mathcal{F}_C \subseteq \mathcal{D}$.
Another Version of the Main Results When $\mathcal{F}$ is Strongly Normal (SN)

Let $\mathcal{F}$ be any finite collection of polyhedra such that every $\mathcal{F}$-\(\cap\) is acyclic. Then for every $P \in \mathcal{F}$, we define: 
\[ \mathcal{N}^*(P, \mathcal{F}) = \{ F \in \mathcal{F} \setminus \{P\} \mid F \cap P \neq \emptyset \} \]

Each member of $\mathcal{N}^*(P, \mathcal{F})$ will be called an $\mathcal{F}$-neighbo

We say $\mathcal{F}$ is strongly normal (SN) if the following is true:

- \( \forall P \in \mathcal{F} \) . $P$ intersects every nonempty intersection of two or more $\mathcal{F}$-neighbors of $P$.

Equivalently:

- \( \forall P \in \mathcal{F} \) . $P$ intersects every $\mathcal{N}^*(P, \mathcal{F})$-\(\cap\).

Motivating Example: Any set of grid cells of a Cartesian grid (of any dimension) is SN.
Another Version of the Main Results When $\mathcal{F}$ is Strongly Normal (SN)

Let $\mathcal{F}$ be any finite collection of polyhedra such that every $\mathcal{F}$-\(\cap\) is acyclic. Then for every $P \in \mathcal{F}$, we define: 
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Each member of $\mathcal{N}^*(P,\mathcal{F})$ will be called an $\mathcal{F}$-neighbor of $P$.

We say $\mathcal{F}$ is **strongly normal** (SN) if the following is true:

- $\forall P \in \mathcal{F} \cdot P$ intersects every nonempty intersection of two or more $\mathcal{F}$-neighbors of $P$.

  Equivalently:  
  $\forall P \in \mathcal{F} \cdot P$ intersects every $\mathcal{N}^*(P,\mathcal{F})$-\(\cap\).

**Motivating Example:** Any set of grid cells of a Cartesian grid (of any dimension) is SN.

- $\mathcal{F}$ is SN $\Rightarrow$ every subcollection of $\mathcal{F}$ is strongly normal
- $\mathcal{F}$ is SN $\Leftrightarrow$ $\mathcal{F}$ is a Helly family of order 2
- $\mathcal{F}$ is SN $\Leftrightarrow$ the collection of all $\mathcal{F}$-\(\cap\)s is strongly normal

SN collections were studied in several papers (1998 – 2007) by Saha, Rosenfeld, and others (Majumder, Brass, Kong).

If $\mathcal{F}$ is SN, we can state Main Theorems 1 and 2 in terms of "cliques" in $\mathcal{F}$. 
A \textit{Non}-Strongly Normal Collection, and a Strongly Normal Collection


In (a), $\mathcal{F} = \{P, Q_1, ..., Q_7, R\}$ is \textit{not} a strongly normal collection.  

Reason:

In (b), $\{P, Q_1, ..., Q_7, R\}$ is a strongly normal collection.
A *Non*-Strongly Normal Collection, and a Strongly Normal Collection


In (a), $\mathcal{F} = \{P, Q_1, ..., Q_7, R\}$ is *not* a strongly normal collection.

**Reason:** $Q_1$ and $Q_2$ are $\mathcal{F}$-neighbors of $P$, $Q_1 \cap Q_2 \neq \emptyset$, but $P \cap Q_1 \cap Q_2 = \emptyset$.

In (b), $\{P, Q_1, ..., Q_7, R\}$ is a strongly normal collection.
In (a), \( \mathcal{F} = \{P, Q_1, \ldots, Q_7, R\} \) is **not** a strongly normal collection.

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In (a), $\mathcal{F} = \{P, Q_1, ..., Q_7, R\}$ is \textbf{not} a strongly normal collection.

Reason: $Q_1$ and $Q_2$ are $\mathcal{F}$-neighbors of $P$, $Q_1 \cap Q_2 \neq \emptyset$, but $P \cap Q_1 \cap Q_2 = \emptyset$.

Another example: Let $\mathcal{F} = \{F_0, ..., F_n\}$ be the set of all $(n-1)$-dimensional faces of an $n$-dimensional simplex. Then $\mathcal{F}$ is \textbf{not} strongly normal (since $F_1, ..., F_n$ are $\mathcal{F}$-neighbors of $F_0$, $F_1 \cap ... \cap F_n \neq \emptyset$, but $F_0 \cap F_1 \cap ... \cap F_n = \emptyset$).
When $\mathcal{F}$ is a good collection of polyhedra that is strongly normal, there is an alternative characterization of $\mathcal{F}$-homology-critical $\mathcal{F}$-\bigcap s:

**Lemma:** Let $\mathcal{F}$ be any strongly normal finite set of polyhedra such that every $\mathcal{F}$-\bigcap is acyclic, and let $C$ be any $\mathcal{F}$-\bigcap. Then $C$ is $\mathcal{F}$-homology-critical just if $\bigcup \{F \in \mathcal{F} \setminus \mathcal{F}_C \mid F$ intersects each member of $\mathcal{F}_C \}$ is not acyclic.
When $\mathcal{F}$ is a good collection of polyhedra that is strongly normal, there is an alternative characterization of $\mathcal{F}$-homology-critical $\mathcal{F}$-\text{\textgreek{f}}s:

**Lemma:** Let $\mathcal{F}$ be any strongly normal finite set of polyhedra such that every $\mathcal{F}$-\text{\textgreek{f}} is acyclic, and let $C$ be any $\mathcal{F}$-\text{\textgreek{f}}. Then $C$ is $\mathcal{F}$-homology-critical just if $\bigcup\{F \in \mathcal{F} \setminus \mathcal{F}_C \mid F$ intersects each member of $\mathcal{F}_C\}$ is **not** acyclic.

Recall that, if $Q$ is an inclusion-maximal member of $\mathcal{F}$ (so $\mathcal{F}_Q = \{Q\}$), then

$Q$ is homology-simple in $\mathcal{F} \iff Q$ is **not** $\mathcal{F}$-homology-critical

So on putting $C = \text{any such } Q$ in the above lemma we deduce:
When \( \mathcal{F} \) is a good collection of polyhedra that is strongly normal, there is an alternative characterization of \( \mathcal{F} \)-homology-critical \( \mathcal{F} \)-\( \cap \)s:

**Lemma:** Let \( \mathcal{F} \) be any strongly normal finite set of polyhedra such that every \( \mathcal{F} \)-\( \cap \) is acyclic, and let \( C \) be any \( \mathcal{F} \)-\( \cap \). Then \( C \) is \( \mathcal{F} \)-homology-critical just if \( \bigcup \{ F \in \mathcal{F} \setminus \mathcal{F}_C \mid F \) intersects each member of \( \mathcal{F}_C \} \) is not acyclic.

Recall that, if \( Q \) is an inclusion-maximal member of \( \mathcal{F} \) (so \( \mathcal{F}_Q = \{ Q \} \)), then

\[ Q \text{ is homology-simple in } \mathcal{F} \iff Q \text{ is not } \mathcal{F} \text{-homology-critical} \]

So on putting \( C = \text{ any such } Q \) in the above lemma we deduce:

**Cor.:** Let \( \mathcal{F} \) be any strongly normal finite set of polyhedra such that every \( \mathcal{F} \)-\( \cap \) is acyclic, and let \( Q \) be any inclusion-maximal member of \( \mathcal{F} \). Then \( Q \) is \( \mathcal{F} \)-homology-simple just if \( \bigcup \{ F \in \mathcal{F} \setminus \{ Q \} \mid F \) intersects \( Q \} \) is acyclic.

But this is false when \( \mathcal{F} \) is not strongly normal:
When $\mathcal{F}$ is a good collection of polyhedra that is strongly normal, there is an alternative characterization of $\mathcal{F}$-homology-critical $\mathcal{F}$-$\cap$s:

**Lemma:** Let $\mathcal{F}$ be any **strongly normal** finite set of polyhedra such that every $\mathcal{F}$-$\cap$ is acyclic, and let $C$ be any $\mathcal{F}$-$\cap$. Then $C$ is $\mathcal{F}$-homology-critical just if $\bigcup \{ F \in \mathcal{F} \setminus \mathcal{F}_C \mid F \text{ intersects each member of } \mathcal{F}_C \}$ is **not** acyclic.

Recall that, if $Q$ is an inclusion-maximal member of $\mathcal{F}$ (so $\mathcal{F}_Q = \{Q\}$), then

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**Cor.:** Let $\mathcal{F}$ be any **strongly normal** finite set of polyhedra such that every $\mathcal{F}$-$\cap$ is acyclic, and let $Q$ be any inclusion-maximal member of $\mathcal{F}$. Then $Q$ is $\mathcal{F}$-homology-simple just if $\bigcup \{ F \in \mathcal{F} \setminus \{Q\} \mid F \text{ intersects } Q \}$ is acyclic.

But this is **false** when $\mathcal{F}$ is **not** strongly normal:

If $\mathcal{F} = \{Q_1, \ldots, Q_7, P, R\}$, then $Q_1$ is **not** $\mathcal{F}$-homology-simple as $\text{Attach}(Q_1, \mathcal{F}) = \text{Core}_\mathcal{F}(Q_1)$ is disconnected, but $\bigcup \{ F \in \mathcal{F} \setminus \{Q_1\} \mid F \text{ intersects } Q_1 \} = P \cup Q_2$ is acyclic.
Recall:  **THEOREM:** Let $\mathcal{F}$ be any finite set of polyhedra such that every $\mathcal{F}\cap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then:

(i) $Q$ is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$  if and only if  
there is no $\mathcal{F}$-homology-critical $\mathcal{F}\cap C$ such that $Q \in \mathcal{F}_C \subseteq \mathcal{D}$.

(ii) $\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$  if and only if  
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**Lemma:** Let $\mathcal{F}$ be any **strongly normal** finite set of polyhedra such that every $\mathcal{F}\cap$ is acyclic, and let $C$ be any $\mathcal{F}\cap$. Then $C$ is $\mathcal{F}$-homology-critical  if and only if  
$\bigcup \{ F \in \mathcal{F} \setminus \mathcal{F}_C \mid F$ intersects each member of $\mathcal{F}_C \}$ is not acyclic.

**Hence:**

**THEOREM:** Let $\mathcal{F}$ be any **strongly normal** finite set of polyhedra such that every $\mathcal{F}\cap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then:

(i) $Q$ is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$  if and only if  
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Recall: **THEOREM:** Let $\mathcal{F}$ be any finite set of polyhedra such that every $\mathcal{F} \cap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then:

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Following Bertrand & Couprie, we say a set $S$ is an **essential $\mathcal{F}$-clique** if $S$ has the following three properties:
1. $S \subseteq \mathcal{F}$
2. $\cap S \neq \emptyset$
3. $S = \mathcal{F} \cap S$.

**Readily:** $S$ is an essential $\mathcal{F}$-clique $\iff S = \mathcal{F}_C$ for some $\mathcal{F} \cap C$.

Hence the above theorem can be restated as follows:
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Following Bertrand & Couprie, we say a set $S$ is an \textit{essential $\mathcal{F}$-clique} if $S$ has the following three properties:  
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\textbf{Readily:}  \quad $S$ is an essential $\mathcal{F}$-clique $\iff$ $S = \mathcal{F}_C$ for some $\mathcal{F} \cap C$.

Hence the above theorem can be restated as follows:

\textbf{THEOREM:} Let $\mathcal{F}$ be any strongly normal finite set of polyhedra such that every $\mathcal{F} \cap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then:

(i) $Q$ is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$ \textit{if and only if} there is no essential $\mathcal{F}$-clique $S$ such that $Q \in S \subseteq \mathcal{D}$ and $\bigcup \{F \in \mathcal{F} \setminus S \mid F$ intersects each member of $S\}$ is not acyclic.

(ii) $\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$ \textit{if and only if} there is no essential $\mathcal{F}$-clique $S$ such that $S \subseteq \mathcal{D}$ and $\bigcup \{F \in \mathcal{F} \setminus S \mid F$ intersects each member of $S\}$ is not acyclic.

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Following Bertrand & Couprie, we say a set $S$ is an **essential $\mathcal{F}$-clique** if $S$ has the following three properties:  
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**Readily:** $S$ is an essential $\mathcal{F}$-clique $\iff S = \mathcal{F}_C$ for some $\mathcal{F} \cap C$.

Hence the above theorem can be restated as follows:

**THEOREM:** Let $\mathcal{F}$ be any strongly normal finite set of polyhedra such that every $\mathcal{F} \cap$ is acyclic, and let $Q \in D \subseteq \mathcal{F}$. Then:

(i) $Q$ is $\mathbb{P}$-homology-simple for $D$ in $\mathcal{F}$ **if and only if** there is no essential $\mathcal{F}$-clique $S$ such that $Q \in S \subseteq D$ and $\bigcup \{F \in \mathcal{F} \setminus S \mid F$ intersects each member of $S\}$ is not acyclic.

(ii) $D$ is hereditarily homology-simple in $\mathcal{F}$ **if and only if** there is no essential $\mathcal{F}$-clique $S$ such that $S \subseteq D$ and $\bigcup \{F \in \mathcal{F} \setminus S \mid F$ intersects each member of $S\}$ is not acyclic.

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- This result gives local characterizations of $\mathbb{P}$-homology-simpleness and hereditary-homology-simpleness in terms of the "common $\mathcal{F}$-neighbors" of essential $\mathcal{F}$-cliques (instead of cores of $\mathcal{F} \cap S$).
- But it assumes $\mathcal{F}$ is strongly normal (unlike Main Theorems 1 & 2).
- In the case where $\mathcal{F}$ is a set of grid cells of a 3D Cartesian grid, closely related results were found by Bertrand & Couprie (2014).
Recall: **MAIN THEOREM 1** Let $\mathcal{F}$ be a finite set of polyhedra such that every $\mathcal{F}$-$\cap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:

1. $Q$ is $\mathcal{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$.
2. Every $\mathcal{F}$-homology-critical $\mathcal{D}$-$\cap$ contained in $Q$ is also contained in a member of $\mathcal{F} \setminus \mathcal{D}$.

**Proof of the $2 \Rightarrow 1$ Part of Main Theorem 1**

We say a set $C$ of $\mathcal{F}$-$\cap$s is **inclusion-closed** if $C$ satisfies:

- Whenever $X \in C$ and $Y$ is an $\mathcal{F}$-$\cap$ such that $Y \subseteq X$, we have that $Y \in C$.

**Lemma:** Let $C$ be any inclusion-closed set of $\mathcal{F}$-$\cap$s, and let $M$ be any inclusion-maximal member of $C$.

Then $C \setminus \{M\}$ is an inclusion-closed set of $\mathcal{F}$-$\cap$s such that:

1. $M \cap \bigcup(C \setminus \{M\}) = \bigcup\{M \cap Z \mid Z \in C \setminus \{M\}\}$
   
   $= \bigcup\{Y \mid Y$ is an $\mathcal{F}$-$\cap$ and $Y \subsetneq M\} = \text{Core}_\mathcal{F}(M)$

2. If $M$ is not $\mathcal{F}$-homology-critical, then the inclusion of $\bigcup(C \setminus \{M\})$ in $\bigcup C$ induces a homology isomorphism.

Assertion 2 follows from assertion 1, excision, and the exact homology sequences of $(M, M \cap \bigcup(C \setminus \{M\}))$ and $(\bigcup C, \bigcup(C \setminus \{M\}))$. 
Recall: **MAIN THEOREM 1** Let $\mathcal{F}$ be a finite set of polyhedra such that every $\mathcal{F} \cap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:

1. $Q$ is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$.
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**Proof of the $2 \Rightarrow 1$ Part of Main Theorem 1**

We say a set $\mathcal{C}$ of $\mathcal{F} \cap$ s is **inclusion-closed** if $\mathcal{C}$ satisfies:

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**Lemma:** Let $\mathcal{C}$ be any inclusion-closed set of $\mathcal{F} \cap$ s, and let $M$ be any inclusion-maximal member of $\mathcal{C}$. Then $\mathcal{C} \setminus \{M\}$ is an inclusion-closed set of $\mathcal{F} \cap$ s such that:

2. If $M$ is not $\mathcal{F}$-homology-critical, then the inclusion of $\bigcup(\mathcal{C} \setminus \{M\})$ in $\bigcup \mathcal{C}$ induces a homology isomorphism.
Recall: MAIN THEOREM 1  Let $\mathcal{F}$ be a finite set of polyhedra such that every $\mathcal{F}$-$\cap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:
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Proof of the $2 \Rightarrow 1$ Part of Main Theorem 1

We say a set $\mathcal{C}$ of $\mathcal{F}$-$\cap$s is **inclusion-closed** if $\mathcal{C}$ satisfies:
- Whenever $X \in \mathcal{C}$ and $Y$ is an $\mathcal{F}$-$\cap$ such that $Y \subseteq X$, we have that $Y \in \mathcal{C}$.

**Lemma:** Let $\mathcal{C}$ be any inclusion-closed set of $\mathcal{F}$-$\cap$s, and let $M$ be any inclusion-maximal member of $\mathcal{C}$. Then $\mathcal{C} \setminus \{M\}$ is an inclusion-closed set of $\mathcal{F}$-$\cap$s such that:
2. If $M$ is not $\mathcal{F}$-homology-critical, then the inclusion of $\bigcup(\mathcal{C} \setminus \{M\})$ in $\bigcup \mathcal{C}$ induces a homology isomorphism.

**Corollary A1:** Let $\mathcal{A} \supseteq \mathcal{B}$ be any two inclusion-closed sets of $\mathcal{F}$-$\cap$s such that no member of $\mathcal{A} \setminus \mathcal{B}$ is $\mathcal{F}$-homology-critical. Then the inclusion of $\bigcup \mathcal{B}$ in $\bigcup \mathcal{A}$ induces a homology isomorphism.
Proof of the 2 ⇒ 1 Part of Main Theorem 1

We say a set \( C \) of \( \mathcal{F}\cap s \) is **inclusion-closed** if \( C \) satisfies:

- Whenever \( X \in C \) and \( Y \) is an \( \mathcal{F}\cap \) such that \( Y \subseteq X \), we have that \( Y \in C \).

**Lemma:** Let \( C \) be any inclusion-closed set of \( \mathcal{F}\cap s \), and let \( M \) be any inclusion-maximal member of \( C \). Then \( C \setminus \{M\} \) is an inclusion-closed set of \( \mathcal{F}\cap s \) such that:

2. If \( M \) is not \( \mathcal{F}\)-homology-critical, then the inclusion of \( \bigcup(C \setminus \{M\}) \) in \( \bigcup C \) induces a homology isomorphism.

**Corollary A1:** Let \( \mathcal{A} \supseteq \mathcal{B} \) be any two inclusion-closed sets of \( \mathcal{F}\cap s \) such that no member of \( \mathcal{A} \setminus \mathcal{B} \) is \( \mathcal{F}\)-homology-critical. Then the inclusion of \( \bigcup \mathcal{B} \) in \( \bigcup \mathcal{A} \) induces a homology isomorphism.

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Proof of the 2 ⇒ 1 Part of Main Theorem 1

We say a set $\mathcal{C}$ of $\mathcal{F} \cap s$ is **inclusion-closed** if $\mathcal{C}$ satisfies:

- Whenever $X \in \mathcal{C}$ and $Y$ is an $\mathcal{F} \cap$ such that $Y \subseteq X$, we have that $Y \in \mathcal{C}$.

**Lemma:** Let $\mathcal{C}$ be any inclusion-closed set of $\mathcal{F} \cap s$, and let $M$ be any inclusion-maximal member of $\mathcal{C}$.

Then $\mathcal{C} \setminus \{M\}$ is an inclusion-closed set of $\mathcal{F} \cap s$ such that:

2. If $M$ is not $\mathcal{F}$-homology-critical, then the inclusion of $\bigcup (\mathcal{C} \setminus \{M\})$ in $\bigcup \mathcal{C}$ induces a homology isomorphism.

**Corollary A1:** Let $\mathcal{A} \supseteq \mathcal{B}$ be any two inclusion-closed sets of $\mathcal{F} \cap s$ such that no member of $\mathcal{A} \setminus \mathcal{B}$ is $\mathcal{F}$-homology-critical. Then the inclusion of $\bigcup \mathcal{B}$ in $\bigcup \mathcal{A}$ induces a homology isomorphism.

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**Proof:** Let $\mathcal{C}_0 \supseteq \ldots \supseteq \mathcal{C}_k$ be the sets of $\mathcal{F} \cap s$ defined by $\mathcal{C}_0 = \mathcal{A}$, $\mathcal{C}_k = \mathcal{B}$, and $\mathcal{C}_{j+1} = \mathcal{C}_j \setminus \{M_j\}$, $M_j$ an inclusion-maximal member of $\mathcal{C}_j \setminus \mathcal{B}$, for $0 \leq j < k$.

Here each $M_j$ is also inclusion-maximal in $\mathcal{C}_j$, as $\mathcal{B}$ is inclusion-closed. So, for each $j$, the inclusion of $\bigcup \mathcal{C}_{j+1}$ in $\bigcup \mathcal{C}_j$ induces a homology isomorphism (by Lemma). Hence so does the inclusion of $\bigcup \mathcal{C}_k = \bigcup \mathcal{B}$ in $\bigcup \mathcal{C}_0 = \bigcup \mathcal{A}$. //
Recall: **Corollary A1**: Let \( \mathcal{A} \supseteq \mathcal{B} \) be any two inclusion-closed sets of \( \mathcal{F}\cap s \) such that no member of \( \mathcal{A} \setminus \mathcal{B} \) is \( \mathcal{F} \)-homology-critical. Then the inclusion of \( \bigcup \mathcal{B} \) in \( \bigcup \mathcal{A} \) induces a homology isomorphism.

**MAIN THEOREM 1** Let \( \mathcal{F} \) be a finite set of polyhedra such that every \( \mathcal{F}\cap \) is acyclic, and let \( Q \in \mathcal{D} \subseteq \mathcal{F} \). Then the following are equivalent:
1. \( Q \) is \( \mathbb{P} \)-homology-simple for \( \mathcal{D} \) in \( \mathcal{F} \).
2. Every \( \mathcal{F} \)-homology-critical \( \mathcal{D}\cap \) contained in \( Q \) is also contained in a member of \( \mathcal{F} \setminus \mathcal{D} \).

Completion of the Proof of the 2 ⇒ 1 Part of Main Theorem 1
Under the hypotheses of Main Thm. 1, let \( \mathcal{S} \) be any subset of \( \mathcal{D} \setminus \{Q\} \), let \( \mathcal{A} = \) set of \( \mathcal{F}\cap \)s that lie in at least one member of \( \mathcal{F} \setminus \mathcal{S} \) and let \( \mathcal{B} = \) set of \( \mathcal{F}\cap \)s that lie in at least one member of \( (\mathcal{F} \setminus \mathcal{S}) \setminus \{Q\} \). So \( \mathcal{A} \setminus \mathcal{B} = \) set of \( \mathcal{F}\cap \)s that lie in \( Q \) but not in any member of \( (\mathcal{F} \setminus \mathcal{S}) \setminus \{Q\} \).

Now:

condition 2 of Main Thm. 1
\[ \Rightarrow \text{every } \mathcal{F}\text{-homology-critical } \mathcal{F}\cap \text{ that lies in } Q \text{ also lies in a member of } \mathcal{F} \setminus \mathcal{D} \subseteq (\mathcal{F} \setminus \mathcal{S}) \setminus \{Q\} \]
\[ \Rightarrow \text{no member of } \mathcal{A} \setminus \mathcal{B} \text{ is } \mathcal{F}\text{-homology-critical} \]
\[ \Rightarrow Q \text{ is homology-simple in } \mathcal{F} \setminus \mathcal{S} \text{ (by Cor. A1)} \]
\[ \Rightarrow \text{condition 1 of Main Thm. 1 (as } \mathcal{S} \text{ is an arbitrary subset of } \mathcal{D} \setminus \{Q\}). \&/}
Proof of the 1 \Rightarrow 2 Part of Main Theorem 1: A Preliminary Lemma

Recall: Whenever \( D \subseteq \mathcal{F} \) and \( C \) is a \( D \)-\( \cap \), we define \( \mathcal{D}_C \triangleq \{ D \in \mathcal{D} \mid C \subseteq D \} \)

Lemma 5: Suppose condition 2 is not satisfied. Let \( Q \in \mathcal{D} \subseteq \mathcal{F} \) and let \( C \) be an \( \mathcal{F} \)-homology-critical \( D \)-\( \cap \) that is contained in \( Q \) but not contained in any member of \( \mathcal{F} \setminus \mathcal{D} \). Then the set \( (\bigcap \mathcal{D}_C) \cap \text{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C) = C \cap \text{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C) = C \cap \bigcup (\mathcal{F} \setminus \mathcal{D}_C) \) is the \( \mathcal{F} \)-core of \( C \) and is not acyclic.

Proof: As \( C \) is not contained in any member of \( \mathcal{F} \setminus \mathcal{D} \), we have that \( \mathcal{D}_C = \mathcal{F}_C \) and hence \( \mathcal{F} \setminus \mathcal{D}_C = \mathcal{F} \setminus \mathcal{F}_C = \{ F \in \mathcal{F} \mid F \not\supseteq C \} \)
Thus \( C \cap \bigcup (\mathcal{F} \setminus \mathcal{D}_C) = C \cap \bigcup \{ F \in \mathcal{F} \mid F \not\supseteq C \} = \text{Core}_\mathcal{F}(C) \), which is not acyclic as \( C \) is \( \mathcal{F} \)-homology-critical. //
Proof of the \(1 \Rightarrow 2\) Part of Main Theorem 1: A Preliminary Lemma

Recall: Whenever \(\mathcal{D} \subseteq \mathcal{F}\) and \(C\) is a \(\mathcal{D}\)-\(\cap\), we define \(\mathcal{D}_C \overset{\text{def}}{=} \{D \in \mathcal{D} \mid C \subseteq D\}\)

**Lemma 5:** Suppose condition \(2\) is **not** satisfied. Let \(Q \in \mathcal{D} \subseteq \mathcal{F}\) and let \(C\) be an \(\mathcal{F}\)-homology-critical \(\mathcal{D}\)-\(\cap\) that is contained in \(Q\) but **not** contained in any member of \(\mathcal{F} \setminus \mathcal{D}\). Then the set\[
(\bigcap \mathcal{D}_C) \cap \text{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C) = C \cap \text{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C) = C \cap \bigcup (\mathcal{F} \setminus \mathcal{D}_C)
\]
is the \(\mathcal{F}\)-core of \(C\) and is **not** acyclic.

**Proof:** As \(C\) is not contained in any member of \(\mathcal{F} \setminus \mathcal{D}\), we have that \(\mathcal{D}_C = \mathcal{F}_C\) and hence \(\mathcal{F} \setminus \mathcal{D}_C = \mathcal{F} \setminus \mathcal{F}_C = \{F \in \mathcal{F} \mid F \not\supset C\}\). Thus \(C \cap \bigcup (\mathcal{F} \setminus \mathcal{D}_C) = C \cap \bigcup \{F \in \mathcal{F} \mid F \not\supset C\} = \text{Core}_\mathcal{F}(C),\) which is not acyclic as \(C\) is \(\mathcal{F}\)-homology-critical. //

Also recall: **Corollary 2:** Let \(Q \in \mathcal{D} \subseteq \mathcal{F}\). Then \(Q\) is \(\mathcal{P}\)-homology-simple for \(\mathcal{D}\) in \(\mathcal{F}\) if and only if \(\text{Attach}(Q, \mathcal{F} \setminus S)\) is acyclic for all \(S \subseteq \mathcal{D}\).

We now prove \(\text{not } 2 \Rightarrow \text{not } 1\) by showing (for \(Q \in \mathcal{D} \subseteq \mathcal{F}\)) that:

If \((\bigcap \mathcal{D}_C) \cap \text{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C)\) is **not** acyclic, then it is **not** true that \("\text{Attach}(Q, \mathcal{F} \setminus S)\) is acyclic for all \(S \subseteq \mathcal{D}\)."
Proof of the $1 \Rightarrow 2$ Part of Main Theorem 1: Properties of Acyclic Polyhedra

**Property A:** Let $S$ and $T$ be polyhedra that satisfy any two of the following conditions. Then $S$ and $T$ satisfy all three conditions:

1. Each of $S$ and $T$ is acyclic.
2. $S \cap T$ is acyclic.
3. $S \cup T$ is acyclic.

Property A follows from a standard result of algebraic topology—the Mayer-Vietoris exact sequence for reduced homology of polyhedra:

\[
\cdots \to \tilde{H}_p(S \cap T) \to \tilde{H}_p(S) \oplus \tilde{H}_p(T) \to \tilde{H}_p(S \cup T) \to \tilde{H}_{p-1}(S \cap T) \to \tilde{H}_{p-1}(S) \oplus \tilde{H}_{p-1}(T) \to \cdots
\]

**Property B:** Let $\mathcal{P}$ be a finite collection of polyhedra. Then the following are equivalent:

(i) Every nonempty subcollection of $\mathcal{P}$ has an acyclic intersection:

$\bigcap \mathcal{P}'$ is acyclic whenever $\emptyset \neq \mathcal{P}' \subseteq \mathcal{P}$.

(ii) Every nonempty subcollection of $\mathcal{P}$ has an acyclic union:

$\bigcup \mathcal{P}'$ is acyclic whenever $\emptyset \neq \mathcal{P}' \subseteq \mathcal{P}$.

Property B follows from Property A by induction on the collection's size.
Recall: **MAIN THEOREM 1** Let $\mathcal{F}$ be a finite set of polyhedra such that every $\mathcal{F} \cap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:

1. $Q$ is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$.
2. Every $\mathcal{F}$-homology-critical $\mathcal{D} \cap$ contained in $Q$ is also contained in a member of $\mathcal{F} \setminus \mathcal{D}$.

**Notation:** If $C$ is any $\mathcal{D} \cap$, then: $\mathcal{D}_C \overset{\text{def}}{=} \{D \in \mathcal{D} \mid C \subseteq D\}$

**Lemma 5:** Suppose condition 2 is not satisfied. Let $Q \in \mathcal{D} \subseteq \mathcal{F}$ and let $C$ be an $\mathcal{F}$-homology-critical $\mathcal{D} \cap$ that is contained in $Q$ but not contained in any member of $\mathcal{F} \setminus \mathcal{D}$. Then:

\[
(\cap \mathcal{D}_C) \cap \text{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C) \text{ is the } \mathcal{F}\text{-core of } C \text{ and is not acyclic.}
\]

**Proof of the $1 \Rightarrow 2$ Part of Main Theorem 1**

Suppose 2 is not satisfied. Then $\exists$ an $\mathcal{F}$-homology-critical $\mathcal{D} \cap$, $C$, such that $C$ is contained in $Q$ but $C$ is not contained in any member of $\mathcal{F} \setminus \mathcal{D}$.

We will deduce that $Q$ is not $\mathbb{P}$-homology-simple for $\mathcal{D}_C$ in $\mathcal{F}$, which implies 1 is also not satisfied. To do this, we first note that:

\[
\begin{align*}
\cap(\{\text{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C)\} \cup \{Q \cap D \mid D \in \mathcal{D}_C\}) \\
= \text{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C) \cap \cap \mathcal{D}_C & \text{ is not acyclic, by Lemma 5.}
\end{align*}
\]
Proof of the 1 $\Rightarrow$ 2 Part of Main Theorem 1

Suppose 2 is not satisfied. Then $\exists$ an $\mathcal{F}$-homology-critical $\mathcal{D}\cap$, $C$, such that $C$ is contained in $Q$ but $C$ is not contained in any member of $\mathcal{F}\setminus\mathcal{D}$. We will deduce that $Q$ is not $\mathbb{P}$-homology-simple for $\mathcal{D}_C$ in $\mathcal{F}$, which implies 1 is also not satisfied. To do this, we first note that:

(a) $\bigcap(\{\text{Attach}(Q,\mathcal{F}\setminus\mathcal{D}_C)\} \cup \{Q\cap D \mid D \in \mathcal{D}_C\})$

$= \text{Attach}(Q,\mathcal{F}\setminus\mathcal{D}_C) \cap \bigcap \mathcal{D}_C$ is not acyclic, by Lemma 5.
Proof of the $1 \Rightarrow 2$ Part of Main Theorem 1

Suppose $2$ is not satisfied. Then $\exists$ an $\mathcal{F}$-homology-critical $\mathcal{D} \cap$, $C$, such that $C$ is contained in $Q$ but $C$ is not contained in any member of $\mathcal{F} \setminus \mathcal{D}$.

We will deduce that $Q$ is not $\mathbb{P}$-homology-simple for $\mathcal{D}_C$ in $\mathcal{F}$, which implies $1$ is also not satisfied. To do this, we first note that:

(a) $\bigcap(\{\text{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C)\} \cup \{Q \cap D \mid D \in \mathcal{D}_C\})$ is not acyclic, by Lemma 5.
Proof of the $1 \Rightarrow 2$ Part of Main Theorem 1

Suppose 2 is not satisfied. Then $\exists$ an $\mathcal{F}$-homology-critical $\mathcal{D} \cap$, $C$, such that $C$ is contained in $Q$ but $C$ is not contained in any member of $\mathcal{F} \setminus \mathcal{D}$.

We will deduce that $Q$ is not $\mathbb{P}$-homology-simple for $\mathcal{D}_C$ in $\mathcal{F}$, which implies 1 is also not satisfied. To do this, we first note that:

(a) $\cap(\{\text{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C)\} \cup \{Q \cap D \mid D \in \mathcal{D}_C\})$ is not acyclic, by Lemma 5.

Moreover:

(b) The $\cap$ of any nonempty subcollection of $\{Q \cap D \mid D \in \mathcal{D}_C\}$ is an acyclic superset of $Q \cap \cap \mathcal{D}_C = Q \cap C = C$. 
Proof of the $1 \Rightarrow 2$ Part of Main Theorem 1

Suppose 2 is not satisfied. Then $\exists$ an $\mathcal{F}$-homology-critical $\mathcal{D}-\cap$, $C$, such that $C$ is contained in $Q$ but $C$ is not contained in any member of $\mathcal{F} \setminus \mathcal{D}$. We will deduce that $Q$ is not $\mathcal{P}$-homology-simple for $\mathcal{D}_C$ in $\mathcal{F}$, which implies 1 is also not satisfied. To do this, we first note that:

(a) $\cap(\{\text{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C)\} \cup \{Q \cap D \mid D \in \mathcal{D}_C\})$ is not acyclic, by Lemma 5.

Moreover:

(b) The $\cap$ of any nonempty subcollection of $\{Q \cap D \mid D \in \mathcal{D}_C\}$ is an acyclic superset of $Q \cap \cap \mathcal{D}_C = Q \cap C = C$.

Recall: Property B Let $\mathcal{P}$ be a finite collection of polyhedra. Then the following are equivalent:

(i) Every nonempty subcollection of $\mathcal{P}$ has an acyclic intersection:
(ii) Every nonempty subcollection of $\mathcal{P}$ has an acyclic union:
Proof of the $1 \Rightarrow 2$ Part of Main Theorem 1

Suppose $2$ is not satisfied. Then $\exists$ an $\mathcal{F}$-homology-critical $\mathcal{D} \cap$, $C$, such that $C$ is contained in $Q$ but $C$ is not contained in any member of $\mathcal{F} \setminus \mathcal{D}$. We will deduce that $Q$ is not $\mathbb{P}$-homology-simple for $\mathcal{D}_C$ in $\mathcal{F}$, which implies $1$ is also not satisfied. To do this, we first note that:

(a) $\cap(\{\text{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C)\} \cup \{Q \cap D \mid D \in \mathcal{D}_C\})$ is not acyclic, by Lemma 5.

Moreover:

(b) The $\cap$ of any nonempty subcollection of $\{Q \cap D \mid D \in \mathcal{D}_C\}$ is an acyclic superset of $Q \cap \cap \mathcal{D}_C = Q \cap C = C$.

Recall: **Property B** Let $\mathbb{P}$ be a finite collection of polyhedra. Then the following are equivalent:

(i) Every nonempty subcollection of $\mathbb{P}$ has an acyclic intersection:

(ii) Every nonempty subcollection of $\mathbb{P}$ has an acyclic union:

(b) and Property B imply:

(c) The $\cup$ of any nonempty subcollection of $\{Q \cap D \mid D \in \mathcal{D}_C\}$ is acyclic.
Proof of the $1 \Rightarrow 2$ Part of Main Theorem 1

Suppose 2 is not satisfied. Then $\exists$ an $\mathcal{F}$-homology-critical $\mathcal{D}\setminus\mathcal{C}$, such that $\mathcal{C}$ is contained in $\mathcal{Q}$ but $\mathcal{C}$ is not contained in any member of $\mathcal{F}\setminus\mathcal{D}$.

We will deduce that $\mathcal{Q}$ is not $\mathcal{P}$-homology-simple for $\mathcal{D}_\mathcal{C}$ in $\mathcal{F}$, which implies 1 is also not satisfied. To do this, we first note that:

(a) $\cap$\(\{\text{Attach}(\mathcal{Q}, \mathcal{F}\setminus\mathcal{D}_\mathcal{C})\} \cup \{\mathcal{Q}\cap D \mid D \in \mathcal{D}_\mathcal{C}\}\) is not acyclic, by Lemma 5.

(b) The $\cap$ of any nonempty subcollection of $\{\mathcal{Q}\cap D \mid D \in \mathcal{D}_\mathcal{C}\}$ is an acyclic superset of $\mathcal{Q}\cap\cap\mathcal{D}_\mathcal{C} = \mathcal{Q}\cap\mathcal{C} = \mathcal{C}$.

Recall: Property B Let $\mathcal{P}$ be a finite collection of polyhedra. Then the following are equivalent:

(i) Every nonempty subcollection of $\mathcal{P}$ has an acyclic intersection:
(ii) Every nonempty subcollection of $\mathcal{P}$ has an acyclic union:

(b) and Property B imply:

(c) The $\cup$ of any nonempty subcollection of $\{\mathcal{Q}\cap D \mid D \in \mathcal{D}_\mathcal{C}\}$ is acyclic.
Proof of the $1 \Rightarrow 2$ Part of Main Theorem 1

Suppose 2 is not satisfied. Then $\exists$ an $\mathcal{F}$-homology-critical $\mathcal{D}\cap$, $C$, such that $C$ is contained in $Q$ but $C$ is not contained in any member of $\mathcal{F}\setminus\mathcal{D}$.

We will deduce that $Q$ is not $\mathbb{P}$-homology-simple for $\mathcal{D}_C$ in $\mathcal{F}$, which implies 1 is also not satisfied. To do this, we first note that:

(a) $\bigcap(\{\text{Attach}(Q, \mathcal{F}\setminus\mathcal{D}_C)\} \cup \{Q\cap D \mid D \in \mathcal{D}_C\})$ is not acyclic, by Lemma 5.
(b) The $\bigcap$ of any nonempty subcollection of $\{Q\cap D \mid D \in \mathcal{D}_C\}$ is an acyclic superset of $Q\cap \bigcap \mathcal{D}_C = Q\cap C = C$.

Recall: Property B Let $\mathcal{P}$ be a finite collection of polyhedra. Then the following are equivalent:

(i) Every nonempty subcollection of $\mathcal{P}$ has an acyclic intersection:
(ii) Every nonempty subcollection of $\mathcal{P}$ has an acyclic union:

(b) and Property B imply:
(c) The $\bigcup$ of any nonempty subcollection of $\{Q\cap D \mid D \in \mathcal{D}_C\}$ is acyclic.

BUT, (a) and Property B imply:
(d) $\exists$ a nonempty subcollection of $\{\text{Attach}(Q, \mathcal{F}\setminus\mathcal{D}_C)\} \cup \{Q\cap D \mid D \in \mathcal{D}_C\}$ whose $\bigcup$ is not acyclic.
Proof of the 1 ⇒ 2 Part of Main Theorem 1

Suppose 2 is not satisfied. Then ∃ an \( \mathcal{F} \)-homology-critical \( \mathcal{D} \)-∩, \( C \), such that \( C \) is contained in \( Q \) but \( C \) is not contained in any member of \( \mathcal{F} \setminus \mathcal{D} \).

We will deduce that \( Q \) is not \( \mathbb{P} \)-homology-simple for \( \mathcal{D}_C \) in \( \mathcal{F} \), which implies 1 is also not satisfied.

(a), (b), and Property B imply:

(c) The \( \cup \) of any nonempty subcollection of \( \{ Q \cap D \mid D \in \mathcal{D}_C \} \) is acyclic.

(d) ∃ a nonempty subcollection of \( \{ \text{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C) \} \cup \{ Q \cap D \mid D \in \mathcal{D}_C \} \) whose \( \cup \) is not acyclic.
Proof of the $1 \Rightarrow 2$ Part of Main Theorem 1

Suppose 2 is not satisfied. Then $\exists$ an $\mathcal{F}$-homology-critical $\mathcal{D} \cap$, $C$, such that $C$ is contained in $Q$ but $C$ is not contained in any member of $\mathcal{F} \setminus \mathcal{D}$.

We will deduce that $Q$ is not $\mathbb{P}$-homology-simple for $\mathcal{D}_C$ in $\mathcal{F}$, which implies 1 is also not satisfied.

(a), (b), and Property B imply:

(c) The $\bigcup$ of any nonempty subcollection of $\{Q \cap D \mid D \in \mathcal{D}_C\}$ is acyclic.

(d) $\exists$ a nonempty subcollection of $\{\text{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C)\} \cup \{Q \cap D \mid D \in \mathcal{D}_C\}$ whose $\bigcup$ is not acyclic.

(c) and (d) imply:

(e) $\exists \ T \subseteq \mathcal{D}_C : \text{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C) \cup \bigcup \{Q \cap D \mid D \in T\}$ is not acyclic.

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Proof of the 1 ⇒ 2 Part of Main Theorem 1

Suppose 2 is not satisfied. Then ∃ an $\mathcal{F}$-homology-critical $\mathcal{D}$-$\cap$, $C$, such that $C$ is contained in $Q$ but $C$ is not contained in any member of $\mathcal{F} \setminus \mathcal{D}$. We will deduce that $Q$ is not $\mathbb{P}$-homology-simple for $\mathcal{D}_C$ in $\mathcal{F}$, which implies 1 is also not satisfied.

(a), (b), and Property B imply:

(c) The $\cup$ of any nonempty subcollection of $\{Q \cap D \mid D \in \mathcal{D}_C\}$ is acyclic.

(d) ∃ a nonempty subcollection of $\{\text{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C)\} \cup \{Q \cap D \mid D \in \mathcal{D}_C\}$ whose $\cup$ is not acyclic.

(c) and (d) imply:

(e) ∃ $\mathcal{T} \subseteq \mathcal{D}_C : \text{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C) \cup \cup \{Q \cap D \mid D \in \mathcal{T}\}$ is not acyclic.

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Suppose 2 is not satisfied. Then there exists an $F$-homology-critical $D$-$\cap$, $C$, such that $C$ is contained in $Q$ but $C$ is not contained in any member of $F \setminus D$.

We will deduce that $Q$ is not $\mathbb{P}$-homology-simple for $D_C$ in $F$, which implies 1 is also not satisfied.

(a), (b), and Property B imply:

(c) The union of any nonempty subcollection of $\{Q \cap D \mid D \in D_C\}$ is acyclic.

(d) There exists a nonempty subcollection of $\{\text{Attach}(Q, F \setminus D_C)\} \cup \{Q \cap D \mid D \in D_C\}$ whose union is not acyclic.

(c) and (d) imply:

(e) There exists a subcollection $T \subseteq D_C$ such that $\text{Attach}(Q, F \setminus D_C) \cup \cup \{Q \cap D \mid D \in T\}$ is not acyclic.

(f) For all $T \subseteq D_C$ we have that:

$$\text{Attach}(Q, F \setminus D_C) \cup \cup \{Q \cap D \mid D \in T\} = Q$$ or $\text{Attach}(Q, F \setminus (D_C \setminus T))$ according to whether $Q \in T$ or $Q \notin T$.

As $Q$ is acyclic, (e) and (f) imply:

$$\exists T \subseteq D_C : \text{Attach}(Q, F \setminus (D_C \setminus T))$$ is not acyclic.
Proof of the $1 \Rightarrow 2$ Part of Main Theorem 1

As $Q$ is acyclic, (e) and (f) imply:

$\exists \mathcal{T} \subseteq \mathcal{D}_C : \text{Attach}(Q, \mathcal{F} \setminus (\mathcal{D}_C \setminus \mathcal{T}))$ is not acyclic.

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Proof of the 1 ⇒ 2 Part of Main Theorem 1

As $Q$ is acyclic, (e) and (f) imply:
\[ \exists \mathcal{T} \subseteq \mathcal{D}_C : \text{Attach}(Q, \mathcal{F} \setminus (\mathcal{D}_C \setminus \mathcal{T})) \text{ is not acyclic.} \]

Equivalently:
\[ \exists \mathcal{S} \subseteq \mathcal{D}_C : \text{Attach}(Q, \mathcal{F} \setminus \mathcal{S}) \text{ is not acyclic.} \]

Equivalently (by Corollary 2):
$Q$ is not $\mathbb{P}$-homology-simple for $\mathcal{D}_C$ in $\mathcal{F}$.

So we have shown that 1 is not satisfied. This completes the proof. //

Recall: Corollary 2: Let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then $Q$ is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$ if and only if $\text{Attach}(Q, \mathcal{F} \setminus \mathcal{S})$ is acyclic for all $\mathcal{S} \subseteq \mathcal{D}$.

MAIN THEOREM 1 Let $\mathcal{F}$ be a finite set of polyhedra such that every $\mathcal{F}$-$\bigcap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:
1. $Q$ is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$.
2. Every $\mathcal{F}$-homology-critical $\mathcal{D}$-$\bigcap$ contained in $Q$ is also contained in a member of $\mathcal{F} \setminus \mathcal{D}$.
Summary (1)

• A thinning algorithm simplifies a binary image by reducing its foreground to a thin "skeleton" in a "topology-preserving" way.

• Bertrand's critical kernels have been studied extensively by Bertrand and Couprie, who have used them to design many parallel thinning algorithms that automatically satisfy the requirement of being topology-preserving.

• This talk has presented a variant of the concept of critical kernels: homology-critical kernels. For sets of grid cells of a 2D, 3D, or 4D Cartesian grid, homology-critical and critical are equivalent.

• Many results about critical kernels of such sets become valid for sets of arbitrary convex polytopes of any dimension (and, more generally, sets of arbitrary acyclic polyhedra whose nonempty intersections are acyclic) if they are restated as results about homology-critical kernels.
A 2D example of a collection of polyhedra to which the main results of this talk would apply:

- The polyhedra here are the 2D convex polytopes bounded by the gray lines.
- The green parts of this drawing are irrelevant.
Summary (2)

- One formulation of the requirement that a thinning algorithm be "topology-preserving" is that the set of deleted image elements satisfy the condition of being **homology-simple** in the image foreground $\mathcal{F}$.

- For binary images on grid cells of a 2D, 3D, or 4D Cartesian grid, a fundamental theorem of Bertrand & Couprie (2009) relating to critical kernels provides a useful local necessary and sufficient condition for all subsets of a given set of image elements to be homology-simple in $\mathcal{F}$.

- Main Theorem 2 substitutes **homology-critical** for **critical** in the Bertrand-Couprie theorem, to give an analogous necessary and sufficient condition that is valid for binary images on sets of arbitrary convex polytopes of any dimension (even if some polytopes have overlapping interiors) and, more generally, arbitrary acyclic polyhedra whose nonempty intersections are acyclic.

- When $\mathcal{F}$ is a set of 3D Cartesian grid cells, Bertrand & Couprie (2014) established that their results can be stated in terms of the common neighbors of essential cliques (instead of cores of $\mathcal{F}$-intersections). This is also true of our main results if $\mathcal{F}$ is **strongly normal** (2-Helly).