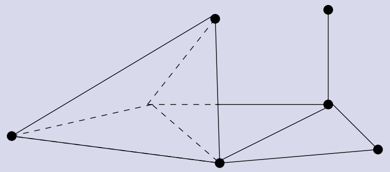
An Axiomatic Approach for Combinatorial Topology

Gilles Bertrand

Université Paris-Est
Laboratoire d'Informatique Gaspard-Monge
Département Informatique et Télécommunications
ESIEE Paris

March 25, 2019

- We investigate a structural (axiomatic) approach related to combinatorial topology and simple homotopy.
- We use completions as a "language" for describing collections of objects.
- We consider objects that are simplicial complexes.



Plan of the presentation

- Motivation
- Simplicial complexes and completions
- Dendrites
- Dyads
- Confluence
- Relative dendrites
- Homotopic pairs
- Conclusion

Motivation

Topological space

A topological space is a set X together with a collection of subsets of X, called open sets and satisfying the following axioms:

- The empty set and X itself are open.
- Any union of open sets is open.
- The intersection of any finite number of open sets is open.

A map $f : X \mapsto Y$ between topological spaces X and Y is called continuous if the inverse image of every open set is open.

Alexandroff spaces and preorders

- When considering finite sets, a topological space is an Alexandroff space, *i.e.*, a topological space in which the intersection of any arbitrary family (not necessarily finite) of open sets is open.
- There is a correspondance between Alexandroff spaces and preorders (binary relations that are reflexive and transitive).
- To any Alexandroff space, we may associate a preorder $\leq$ such that $x \leq y$ if and only if y is contained in all open sets that contain x.
- Conversely, a preorder determines an Alexandroff space: a set O is open for this space if and only if $x \in O$ and $x \leq y$ implies $y \in O$.

Continuous and monotone maps

A map f between two preordered sets X and Y is monotone if $x \leq y$ in X implies $f(x) \leq f(y)$ in Y.

- A map between two preordered sets is monotone if and only if it is a continuous map between the corresponding Alexandroff spaces.
- Conversely, a map between two Alexandroff spaces is continuous if and only if it is a monotone map between the corresponding preordered sets.

Thus, there is a perfect structural equivalence between finite topological spaces and preorders.

Continuous and monotone maps

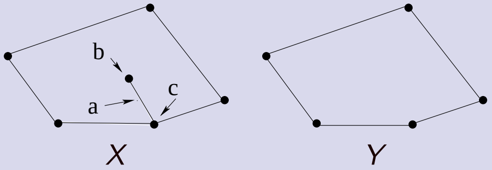
A map f between two preordered sets X and Y is monotone if $x \leq y$ in X implies $f(x) \leq f(y)$ in Y.

- A map between two preordered sets is monotone if and only if it is a continuous map between the corresponding Alexandroff spaces.
- Conversely, a map between two Alexandroff spaces is continuous if and only if it is a monotone map between the corresponding preordered sets.

Thus, there is a perfect structural equivalence between finite topological spaces and preorders.

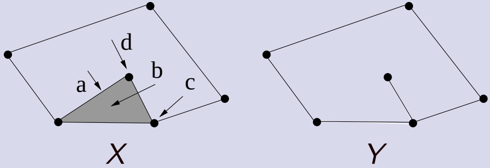
Continuous retraction (1)

Let us consider the two discrete objects X and Y. The object X is made of 6 vertices and 6 edges.
A natural preorder $\leq$ between all these elements is the partial order corresponding to the relation of inclusion between sets.
Thus, we have $b \leq a$ and $c \leq a$.
We see that there exists a monotone map f between X and Y such that f is the identity on all elements of Y, and $f(a) = c$, $f(b) = c$.
Thus, Y corresponds to a continuous retraction of X.



Continuous retraction (2)

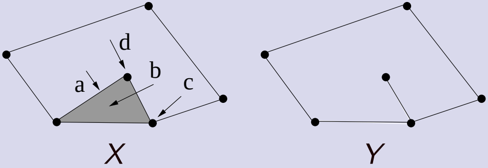
Now, let us consider the following objects X and Y.
We have $d \leq a \leq b$ and $c \leq b$.
We see that it is not possible to build a monotone map f between X and Y, f being the identity on all elements of Y.
For example, if we take $f(a) = c$, $f(b) = c$, we have $d \leq a$, but we have not $f(d) \leq f(a)$.



Thus, in this construction, the classical axioms of topology fail to interpret Y as a continuous retraction of X.

Continuous retraction (2)

Now, let us consider the following objects X and Y.
We have $d \leq a \leq b$ and $c \leq b$.
We see that it is not possible to build a monotone map f between X and Y, f being the identity on all elements of Y.
For example, if we take $f(a) = c$, $f(b) = c$, we have $d \leq a$, but we have not $f(d) \leq f(a)$.

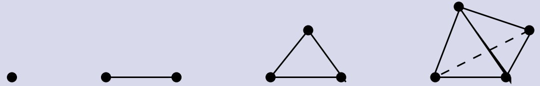


Thus, in this construction, the classical axioms of topology fail to interpret Y as a continuous retraction of X.

Simplicial complexes and completions

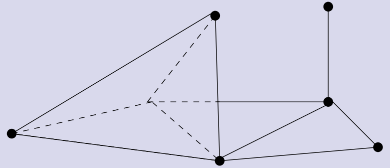
Simplicial complexes

- Let X be a finite family composed of finite sets, X is a simplicial complex if $x \in X$ whenever $x \subseteq y$ and $y \in X$.
- We write $\mathbb{S}$ for the collection of all simplicial complexes.
- Let $X \in \mathbb{S}$. An element of X is a face of X.
- A complex $A \in \mathbb{S}$ is a cell if $A = \emptyset$ or if A has precisely one non-empty maximal face x.
- We write $\mathbb{C}$ for the collection of all cells.



Simplicial complexes

- Let X be a finite family composed of finite sets, X is a simplicial complex if $x \in X$ whenever $x \subseteq y$ and $y \in X$.
- We write $\mathbb{S}$ for the collection of all simplicial complexes.
- Let $X \in \mathbb{S}$. An element of X is a face of X.
- A complex $A \in \mathbb{S}$ is a cell if $A = \emptyset$ or if A has precisely one non-empty maximal face x.
- We write $\mathbb{C}$ for the collection of all cells.



Completions

- A completion may be seen as a rewriting rule which permits to derive collections of objects.
- Completions allows to formulate, in an easy way, inductive definitions.

Completions

- Let $\mathcal{K}$ be an arbitrary sub-collection of $\mathbb{S}$, $\mathcal{K}$ is a dedicated symbol (a kind of variable).
- We say that a property $\langle \mathrm{K} \rangle$ is a completion (on $\mathbb{S}$) if $\langle \mathrm{K} \rangle$ may be expressed as the following property:
  $->$ If $\mathbb{F} \subseteq \mathcal{K}$, then $\mathbb{G} \subseteq \mathcal{K}$ whenever *Cond*$(\mathbb{F}, \mathbb{G})$. $\langle \mathrm{K} \rangle$
  where *Cond*$(\mathbb{F}, \mathbb{G})$ is a condition on a finite collection $\mathbb{F}$ and an arbitrary collection $\mathbb{G}$.
- *Theorem*: Let $\langle \mathrm{K} \rangle$ be a completion on $\mathbb{S}$ and let $\mathbb{X} \subseteq \mathbb{S}$. There exists, under the subset ordering, a unique minimal collection $\mathcal{K}$ which contains $\mathbb{X}$ and which satisfies $\langle \mathrm{K} \rangle$.
- We write $\langle \mathbb{X}; \mathrm{K} \rangle$ for this unique minimal collection.
- If $\langle \mathrm{K} \rangle$ and $\langle \mathrm{Q} \rangle$ are two completions, $\langle \mathrm{K} \rangle \wedge \langle \mathrm{Q} \rangle$ is a completion, the symbol $\wedge$ standing for the logical "and". We write $\langle \mathbb{X}; \mathrm{K}, \mathrm{Q} \rangle$ for $\langle \mathbb{X}; \mathrm{K} \wedge \mathrm{Q} \rangle$.

Completions

- Let $\mathcal{K}$ be an arbitrary sub-collection of $\mathbb{S}$, $\mathcal{K}$ is a dedicated symbol (a kind of variable).
- We say that a property $\langle \mathrm{K} \rangle$ is a completion (on $\mathbb{S}$) if $\langle \mathrm{K} \rangle$ may be expressed as the following property:
  $->$ If $\mathbb{F} \subseteq \mathcal{K}$, then $\mathbb{G} \subseteq \mathcal{K}$ whenever *Cond*$(\mathbb{F}, \mathbb{G})$. $\qquad \langle \mathrm{K} \rangle$
  where *Cond*$(\mathbb{F}, \mathbb{G})$ is a condition on a finite collection $\mathbb{F}$ and an arbitrary collection $\mathbb{G}$.
- *Theorem*: Let $\langle \mathrm{K} \rangle$ be a completion on $\mathbb{S}$ and let $\mathbb{X} \subseteq \mathbb{S}$. There exists, under the subset ordering, a unique minimal collection $\mathcal{K}$ which contains $\mathbb{X}$ and which satisfies $\langle \mathrm{K} \rangle$.
- We write $\langle \mathbb{X}; \mathrm{K} \rangle$ for this unique minimal collection.
- If $\langle \mathrm{K} \rangle$ and $\langle \mathrm{Q} \rangle$ are two completions, $\langle \mathrm{K} \rangle \wedge \langle \mathrm{Q} \rangle$ is a completion, the symbol $\wedge$ standing for the logical "and". We write $\langle \mathbb{X}; \mathrm{K}, \mathrm{Q} \rangle$ for $\langle \mathbb{X}; \mathrm{K} \wedge \mathrm{Q} \rangle$.

Completions

- Let $\mathcal{K}$ be an arbitrary sub-collection of $\mathbb{S}$, $\mathcal{K}$ is a dedicated symbol (a kind of variable).
- We say that a property $\langle \mathrm{K} \rangle$ is a completion (on $\mathbb{S}$) if $\langle \mathrm{K} \rangle$ may be expressed as the following property:
  $->$ If $\mathbb{F} \subseteq \mathcal{K}$, then $\mathbb{G} \subseteq \mathcal{K}$ whenever *Cond*$(\mathbb{F}, \mathbb{G})$. $\langle \mathrm{K} \rangle$
  where *Cond*$(\mathbb{F}, \mathbb{G})$ is a condition on a finite collection $\mathbb{F}$ and an arbitrary collection $\mathbb{G}$.
- *Theorem*: Let $\langle \mathrm{K} \rangle$ be a completion on $\mathbb{S}$ and let $\mathbb{X} \subseteq \mathbb{S}$. There exists, under the subset ordering, a unique minimal collection $\mathcal{K}$ which contains $\mathbb{X}$ and which satisfies $\langle \mathrm{K} \rangle$.
- We write $\langle \mathbb{X}; \mathrm{K} \rangle$ for this unique minimal collection.
- If $\langle \mathrm{K} \rangle$ and $\langle \mathrm{Q} \rangle$ are two completions, $\langle \mathrm{K} \rangle \wedge \langle \mathrm{Q} \rangle$ is a completion, the symbol $\wedge$ standing for the logical "and". We write $\langle \mathbb{X}; \mathrm{K}, \mathrm{Q} \rangle$ for $\langle \mathbb{X}; \mathrm{K} \wedge \mathrm{Q} \rangle$.

Example of a Completion: Connectedness

We define the completion $\langle \Upsilon \rangle$ as follows:

$->$ If $S, T \in \mathbb{K}$, then $S \cup T \in \mathbb{K}$ whenever $S \cap T \neq \{\emptyset\}$. $\langle \Upsilon \rangle$

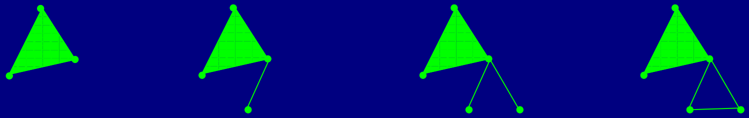
We set $\Pi = \langle \mathbb{C}; \Upsilon \rangle$, Π is precisely the collection of all simplicial complexes which are (path) connected.

Example of a Completion: Connectedness

We define the completion $\langle \Upsilon \rangle$ as follows:

$\Rightarrow$ If $S, T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \neq \{\emptyset\}$. $\langle \Upsilon \rangle$

We set $\Pi = \langle \mathbb{C}; \Upsilon \rangle$, Π is precisely the collection of all simplicial complexes which are (path) connected.

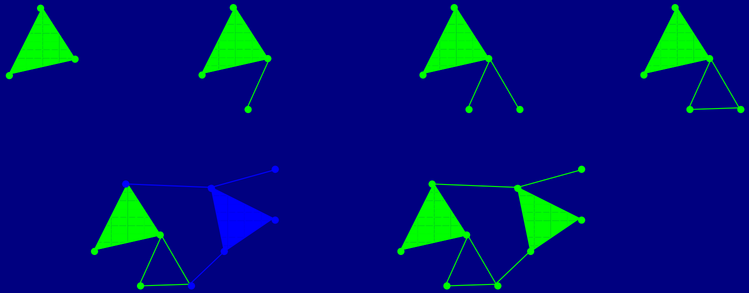


Example of a Completion: Connectedness

We define the completion $\langle \Upsilon \rangle$ as follows:

$\rightarrow$ If $S, T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \neq \{\emptyset\}$. $\langle \Upsilon \rangle$

We set $\Pi = \langle \mathbb{C}; \Upsilon \rangle$, Π is precisely the collection of all simplicial complexes which are (path) connected.



Example of a Completion: Connectedness

We define the completion $\langle \Upsilon \rangle$ as follows:

–> If $S, T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \neq \{\emptyset\}$. $\langle \Upsilon \rangle$

We set $\Pi = \langle \mathbb{C}; \Upsilon \rangle$, Π is precisely the collection of all simplicial complexes which are (path) connected.

We see that this completion is an alternative to the classical definition of connectedness. Furthermore it provides a constructive way for generating all connected complexes.

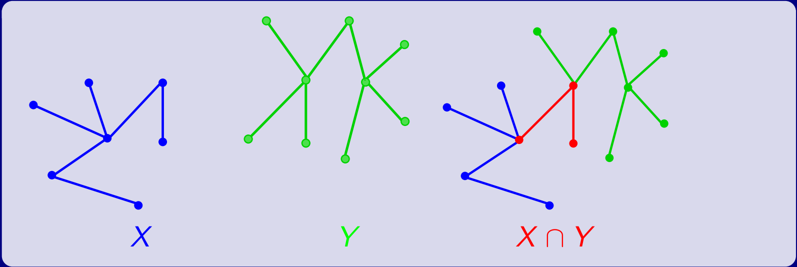
Dendrites

Motivation:

To describe a remarkable collection of acyclic complexes.

Dendrites: the basic idea

Let X and Y be two trees.



$X \cup Y$ is a tree whenever $X \cap Y$ is a tree

$X \cap Y$ is a tree whenever $X \cup Y$ is a tree

Dendrites: the axioms

We define the completions $\langle \mathrm{D1} \rangle$ and $\langle \mathrm{D2} \rangle$ as follows:

For any $S, T \in \mathbb{S}$,

–> If S, $T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \in \mathcal{K}$. $\langle \mathrm{D1} \rangle$

–> If S, $T \in \mathcal{K}$, then $S \cap T \in \mathcal{K}$ whenever $S \cup T \in \mathcal{K}$. $\langle \mathrm{D2} \rangle$

We set $\mathbb{D} = \langle \mathbb{C}; \mathrm{D1}, \mathrm{D2} \rangle$. Each element of $\mathbb{D}$ is a dendrite.

Dendrites: the axioms

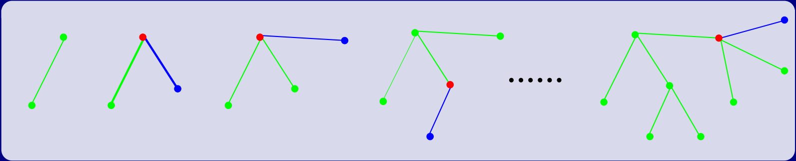
We define the completions $\langle \mathrm{D1} \rangle$ and $\langle \mathrm{D2} \rangle$ as follows:

For any $S, T \in \mathbb{S}$,

-> If S, $T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \in \mathcal{K}$. $\langle \mathrm{D1} \rangle$

-> If S, $T \in \mathcal{K}$, then $S \cap T \in \mathcal{K}$ whenever $S \cup T \in \mathcal{K}$. $\langle \mathrm{D2} \rangle$

We set $\mathbb{D} = \langle \mathbb{C}; \mathrm{D1}, \mathrm{D2} \rangle$. Each element of $\mathbb{D}$ is a dendrite.



Dendrites: the axioms

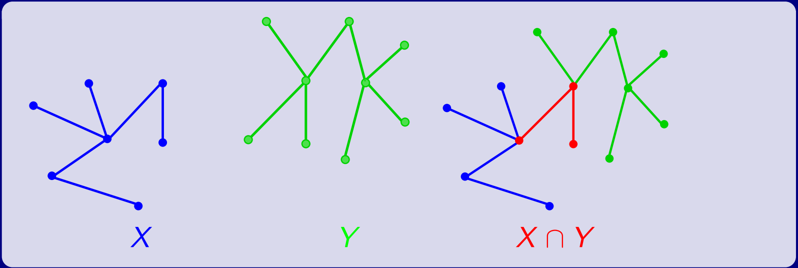
We define the completions $\langle \mathrm{D1} \rangle$ and $\langle \mathrm{D2} \rangle$ as follows:

For any $S, T \in \mathbb{S}$,

–> If S, $T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \in \mathcal{K}$. $\langle \mathrm{D1} \rangle$

–> If S, $T \in \mathcal{K}$, then $S \cap T \in \mathcal{K}$ whenever $S \cup T \in \mathcal{K}$. $\langle \mathrm{D2} \rangle$

We set $\mathbb{D} = \langle \mathbb{C}; \mathrm{D1}, \mathrm{D2} \rangle$. Each element of $\mathbb{D}$ is a dendrite.



Dendrites: the axioms

We define the completions $\langle \mathrm{D1} \rangle$ and $\langle \mathrm{D2} \rangle$ as follows:

For any $S, T \in \mathbb{S}$,

–> If $S,\ T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \in \mathcal{K}$. $\langle \mathrm{D1} \rangle$

–> If $S,\ T \in \mathcal{K}$, then $S \cap T \in \mathcal{K}$ whenever $S \cup T \in \mathcal{K}$. $\langle \mathrm{D2} \rangle$

We set $\mathbb{D} = \langle \mathbb{C}; \mathrm{D1}, \mathrm{D2} \rangle$. Each element of $\mathbb{D}$ is a dendrite.

Let $\mathbb{T}$ denote the collection of all trees. We have :

$$\mathbb{T} = \langle \mathbb{C}[1]; \mathrm{D1}, \mathrm{D2} \rangle$$

Dendrites: the axioms

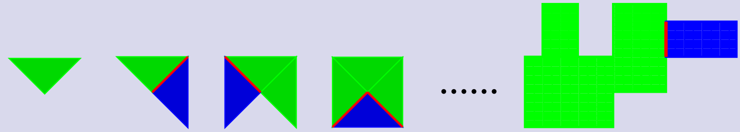
We define the completions $\langle \mathrm{D1} \rangle$ and $\langle \mathrm{D2} \rangle$ as follows:

For any $S, T \in \mathbb{S}$,

–> If $S, T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \in \mathcal{K}$. $\langle \mathrm{D1} \rangle$

–> If $S, T \in \mathcal{K}$, then $S \cap T \in \mathcal{K}$ whenever $S \cup T \in \mathcal{K}$. $\langle \mathrm{D2} \rangle$

We set $\mathbb{D} = \langle \mathbb{C}; \mathrm{D1}, \mathrm{D2} \rangle$. Each element of $\mathbb{D}$ is a dendrite.



Dendrites: the axioms

We define the completions $\langle \mathrm{D1} \rangle$ and $\langle \mathrm{D2} \rangle$ as follows:

For any $S, T \in \mathbb{S}$,

$->$ If S, $T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \in \mathcal{K}$. $\langle \mathrm{D1} \rangle$

$->$ If S, $T \in \mathcal{K}$, then $S \cap T \in \mathcal{K}$ whenever $S \cup T \in \mathcal{K}$. $\langle \mathrm{D2} \rangle$

We set $\mathbb{D} = \langle \mathbb{C}; \mathrm{D1}, \mathrm{D2} \rangle$. Each element of $\mathbb{D}$ is a dendrite.

- A global structure
- A dynamic structure

Dendrites: the axioms

We define the completions $\langle \mathrm{D1} \rangle$ and $\langle \mathrm{D2} \rangle$ as follows:

For any $S, T \in \mathbb{S}$,

−> If S, $T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \in \mathcal{K}$. $\langle \mathrm{D1} \rangle$

−> If S, $T \in \mathcal{K}$, then $S \cap T \in \mathcal{K}$ whenever $S \cup T \in \mathcal{K}$. $\langle \mathrm{D2} \rangle$

We set $\mathbb{D} = \langle \mathbb{C}; \mathrm{D1}, \mathrm{D2} \rangle$. Each element of $\mathbb{D}$ is a dendrite.

- A global structure
- A dynamic structure

Dendrites: the axioms

We define the completions $\langle \mathrm{D1} \rangle$ and $\langle \mathrm{D2} \rangle$ as follows:

For any $S, T \in \mathbb{S}$,

–> If S, $T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \in \mathcal{K}$. $\langle \mathrm{D1} \rangle$

–> If S, $T \in \mathcal{K}$, then $S \cap T \in \mathcal{K}$ whenever $S \cup T \in \mathcal{K}$. $\langle \mathrm{D2} \rangle$

We set $\mathbb{D} = \langle \mathbb{C}; \mathrm{D1}, \mathrm{D2} \rangle$. Each element of $\mathbb{D}$ is a dendrite.

We set $\mathbb{R} = \langle \mathbb{C}; \mathrm{D1} \rangle$. Each element of $\mathbb{R}$ is a ramification. Thus, we have $\mathbb{R} \subseteq \mathbb{D}$.

Let $\mathbb{T}$ denote the collection of all trees. We have :

$$\mathbb{T} = \langle \mathbb{C}[1]; \mathrm{D1}, \mathrm{D2} \rangle = \langle \mathbb{C}[1]; \mathrm{D1} \rangle$$

Dendrites: the axioms

We define the completions $\langle \mathrm{D1} \rangle$ and $\langle \mathrm{D2} \rangle$ as follows:

For any $S, T \in \mathbb{S}$,

−> If S, $T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \in \mathcal{K}$. $\langle \mathrm{D1} \rangle$

−> If S, $T \in \mathcal{K}$, then $S \cap T \in \mathcal{K}$ whenever $S \cup T \in \mathcal{K}$. $\langle \mathrm{D2} \rangle$

We set $\mathbb{D} = \langle \mathbb{C}; \mathrm{D1}, \mathrm{D2} \rangle$. Each element of $\mathbb{D}$ is a dendrite.

We set $\mathbb{R} = \langle \mathbb{C}; \mathrm{D1} \rangle$. Each element of $\mathbb{R}$ is a ramification.
Thus, we have $\mathbb{R} \subseteq \mathbb{D}$.

Let $\mathbb{T}$ denote the collection of all trees. We have :

$$\mathbb{T} = \langle \mathbb{C}[1]; \mathrm{D1}, \mathrm{D2} \rangle = \langle \mathbb{C}[1]; \mathrm{D1} \rangle$$

Dendrites: dendrites and homology

We define the completions $\langle \mathrm{D1} \rangle$ and $\langle \mathrm{D2} \rangle$ as follows:

For any $S, T \in \mathbb{S}$,

–> If $S, T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \in \mathcal{K}$. $\langle \mathrm{D1} \rangle$

–> If $S, T \in \mathcal{K}$, then $S \cap T \in \mathcal{K}$ whenever $S \cup T \in \mathcal{K}$. $\langle \mathrm{D2} \rangle$

We set $\mathbb{D} = \langle \mathbb{C}; \mathrm{D1}, \mathrm{D2} \rangle$. Each element of $\mathbb{D}$ is a dendrite.

It may be shown that a complex is a dendrite if and only if it is acyclic in the sense of integral homology.

Dendrites: dendrites and homology

We define the completions $\langle \mathrm{D1} \rangle$ and $\langle \mathrm{D2} \rangle$ as follows:

For any $S, T \in \mathbb{S}$,

–> If $S, T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \in \mathcal{K}$. $\langle \mathrm{D1} \rangle$

–> If $S, T \in \mathcal{K}$, then $S \cap T \in \mathcal{K}$ whenever $S \cup T \in \mathcal{K}$. $\langle \mathrm{D2} \rangle$

We set $\mathbb{D} = \langle \mathbb{C}; \mathrm{D1}, \mathrm{D2} \rangle$. Each element of $\mathbb{D}$ is a dendrite.

It may be shown that a complex is a dendrite if and only if it is acyclic in the sense of integral homology.

Dyads

Motivation:

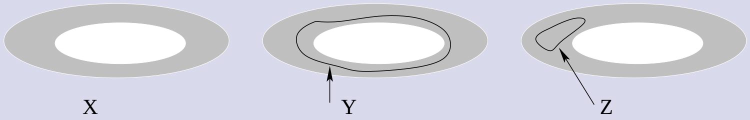
We want to describe collection of arbitrary complexes (complexes that are not necessarily acyclic).

It turns out that the good way to proceed was to consider couple of complexes.

Dyads

Intuitively, a dyad is a couple of complexes (Y, X), with $Y \subseteq X$, such that the cycles of Y are "at the right place with respect to the ones of X".

Three complexes X, Y, and Z, with $Y \subseteq X$ and $Z \subseteq X$:

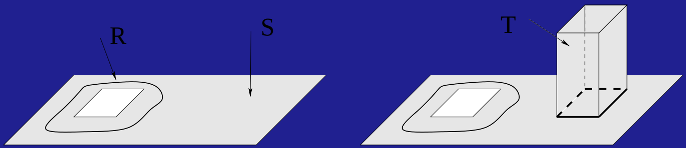


The pair (Y, X) is a dyad, the pair (Z, X) is not a dyad.

Dyads: the basic idea

- We set $\ddot{\mathbb{S}} = \{(Y, X) \mid X, Y \in \mathbb{S}, \text{ with } Y \subseteq X\}$.
- We proceed by considering completions on $\ddot{\mathbb{S}}$. (instead of $\mathbb{S}$)

Three complexes R, S, and T, with $R \subseteq S$:

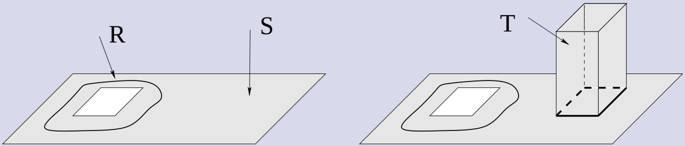


Two objects R, S which constitute a dyad (R, S). An object T which is glued to S. The couple $(S \cap T, T)$ is a dyad, thus $(R, S \cup T)$ is also a dyad.

Dyads: the basic idea

- We set $\ddot{\mathbb{S}} = \{(Y, X) \mid X, Y \in \mathbb{S}, \text{ with } Y \subseteq X\}$.
- We proceed by considering completions on $\ddot{\mathbb{S}}$.
  (instead of $\mathbb{S}$)

Three complexes R, S, and T, with $R \subseteq S$:



Two objects R, S which constitute a dyad (R, S). An object T which is glued to S. The couple $(S \cap T, T)$ is a dyad, thus $(R, S \cup T)$ is also a dyad.

Dyads: the axioms

We set $\ddot{\mathbb{C}} = \{(A, B) \mid A, B \in \mathbb{C}, \text{ with } A \subseteq B\}$.

We define the three completions on $\ddot{\mathbb{S}}$ as follows: For any $(R, S) \in \ddot{\mathbb{S}}$, $T \in \mathbb{S}$,

-> If (R, S) and $(S \cap T, T) \in \ddot{\mathcal{K}}$, then $(R, S \cup T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{X}}1 \rangle$

-> If (R, S) and $(R, S \cup T) \in \ddot{\mathcal{K}}$, then $(S \cap T, T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{X}}2 \rangle$

-> If $(R, S \cup T)$ and $(S \cap T, T) \in \ddot{\mathcal{K}}$, then $(R, S) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{X}}3 \rangle$

We set $\ddot{\mathbb{X}} = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{X}}1, \ddot{\mathrm{X}}2, \ddot{\mathrm{X}}3 \rangle$. Each element of $\ddot{\mathbb{X}}$ is a dyad.

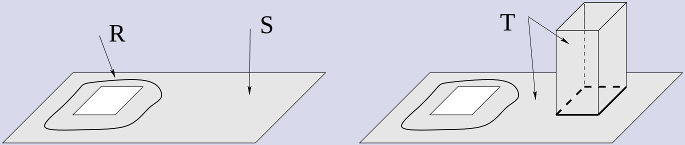
These completions constitute a set of axioms for describing couple of complexes which have "the same topology" and which are "at the right place with respect to each other".

Confluence

Motivation:
We want to describe a fundamental structure of dyads (some fundamental relationships).

Confluence: the basic idea

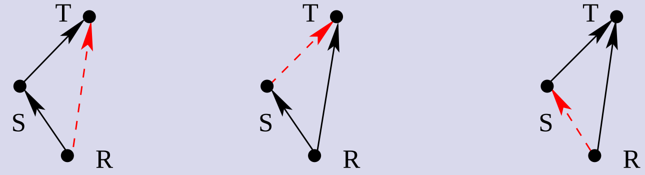
Three complexes R, S, and T, with $R \subseteq S \subseteq T$:



A structural feature of dyads:
If (R, S) and (S, T) are dyads, then (R, T) is a dyad.
(Transitivity)

Confluence: axioms (1)

Three complexes R, S, and T, with $R \subseteq S \subseteq T$:



Transitivity | Upper confluence | Lower confluence

We define the three completions on $\ddot{\mathbb{S}}$ as follows:
For any $(R, S), (S, T), (R, T) \in \ddot{\mathbb{S}}$,

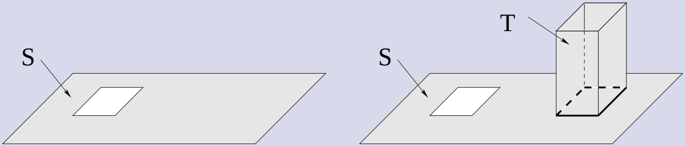
–> If $(R, S) \in \ddot{\mathcal{K}}$ and $(S, T) \in \ddot{\mathcal{K}}$, then $(R, T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{T}} \rangle$
–> If $(R, S) \in \ddot{\mathcal{K}}$ and $(R, T) \in \ddot{\mathcal{K}}$, then $(S, T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{U}} \rangle$
–> If $(R, T) \in \ddot{\mathcal{K}}$ and $(S, T) \in \ddot{\mathcal{K}}$, then $(R, S) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{L}} \rangle$

Confluence: axioms (2)

We define the two completions on $\ddot{\mathbb{S}}$ as follows:
For any $S,\ T \in \mathbb{S}$,

–> If $(S \cap T, T) \in \ddot{\mathcal{K}}$, then $(S, S \cup T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{Y}}1 \rangle$

–> If $(S, S \cup T) \in \ddot{\mathcal{K}}$, then $(S \cap T, T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{Y}}2 \rangle$



Confluence: the confluence theorem

Theorem: We have $\ddot{\mathbb{X}} = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{Y}}2, \ddot{\mathrm{T}}, \ddot{\mathrm{U}}, \ddot{\mathrm{L}} \rangle$.

This theorem provides another way to generate the collection of all dyads. Furthermore it shows the importance of the structural relations $\langle \ddot{\mathrm{T}} \rangle$, $\langle \ddot{\mathrm{U}} \rangle$, and $\langle \ddot{\mathrm{L}} \rangle$.

Confluence: the confluence theorem

Theorem: We have $\ddot{\mathbb{X}} = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{Y}}2, \ddot{\mathrm{T}}, \ddot{\mathrm{U}}, \ddot{\mathrm{L}} \rangle$.

This theorem provides another way to generate the collection of all dyads. Furthermore it shows the importance of the structural relations $\langle \ddot{\mathrm{T}} \rangle$, $\langle \ddot{\mathrm{U}} \rangle$, and $\langle \ddot{\mathrm{L}} \rangle$.

Relative dendrites

Motivation:

We want to establish a link between dyads and dendrites.

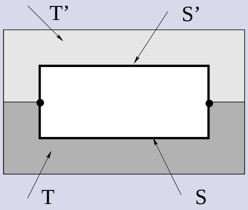
Relative dendrites: relative dendrites

We define two completions on $\ddot{\mathbb{S}}$: For any $(S, T), (S', T') \in \ddot{\mathbb{S}}$,

-> If $(S, T), (S', T'), (S \cap S', T \cap T') \in \ddot{\mathcal{K}}$, then $(S \cup S', T \cup T') \in \ddot{\mathcal{K}}$. $\langle \ddot{Z}1 \rangle$

-> If $(S, T), (S', T'), (S \cup S', T \cup T') \in \ddot{\mathcal{K}}$, then $(S \cap S', T \cap T') \in \ddot{\mathcal{K}}$. $\langle \ddot{Z}2 \rangle$

Each element of $\langle \ddot{\mathbb{C}}^+; \ddot{Z}1, \ddot{Z}2 \rangle$ is called a relative dendrite.



Relative dendrites: the relative dendrites theorem

We define two completions on $\ddot{\mathbb{S}}$: For any $(S, T), (S', T') \in \ddot{\mathbb{S}}$,

-> If $(S, T), (S', T'), (S \cap S', T \cap T') \in \ddot{\mathcal{K}}$, then $(S \cup S', T \cup T') \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{Z}}1 \rangle$

-> If $(S, T), (S', T'), (S \cup S', T \cup T') \in \ddot{\mathcal{K}}$, then $(S \cap S', T \cap T') \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{Z}}2 \rangle$

Each element of $\langle \ddot{\mathbb{C}}^+; \ddot{\mathrm{Z}}1, \ddot{\mathrm{Z}}2 \rangle$ is called a relative dendrite.

Theorem: We have $\ddot{\mathbb{X}} = \langle \ddot{\mathbb{C}}^+; \ddot{\mathrm{Z}}1, \ddot{\mathrm{Z}}2 \rangle$.
In other words a complex is a dyad if and only if it is a relative dendrite.

This theorem provides a third way to generate the collection of all dyads. Furthermore, it allows to establish the forthcoming cancelation theorem.

Relative dendrites: the relative dendrites theorem

We define two completions on $\ddot{\mathbb{S}}$: For any $(S, T), (S', T') \in \ddot{\mathbb{S}}$,

-> If $(S, T), (S', T'), (S \cap S', T \cap T') \in \ddot{\mathcal{K}}$, then $(S \cup S', T \cup T') \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{Z}}1 \rangle$

-> If $(S, T), (S', T'), (S \cup S', T \cup T') \in \ddot{\mathcal{K}}$, then $(S \cap S', T \cap T') \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{Z}}2 \rangle$

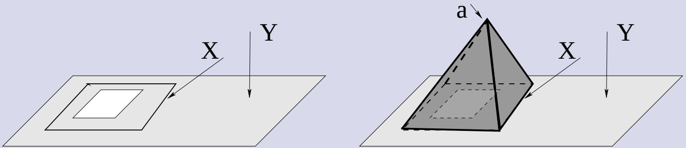
Each element of $\langle \ddot{\mathbb{C}}^+; \ddot{\mathrm{Z}}1, \ddot{\mathrm{Z}}2 \rangle$ is called a relative dendrite.

Theorem: We have $\ddot{\mathbb{X}} = \langle \ddot{\mathbb{C}}^+; \ddot{\mathrm{Z}}1, \ddot{\mathrm{Z}}2 \rangle$.
In other words a complex is a dyad if and only if it is a relative dendrite.

This theorem provides a third way to generate the collection of all dyads. Furthermore, it allows to establish the forthcoming cancelation theorem.

Relative dendrites: the cancelation theorem

A couple (X, Y) which is a dyad, and a cone aX:

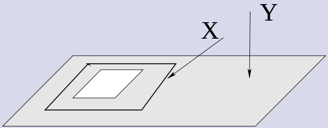


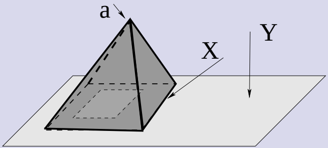
Theorem: Let $(X, Y) \in \ddot{\mathbb{S}}$. The couple (X, Y) is a dyad if and only if $aX \cup Y$ is a dendrite.

Intuitively, this theorem asserts that, if (X, Y) is a dyad, then we cancel out all cycles of Y (*i.e.*, we obtain an acyclic complex), whenever we cancel out those of X (by the way of a cone). Furthermore, it asserts that, if we are able to cancel all cycles of Y by such a way, then (X, Y) is a dyad.

Relative dendrites: the cancelation theorem

A couple (X, Y) which is a dyad, and a cone aX:



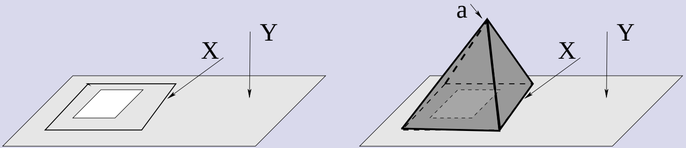


Theorem: Let $(X, Y) \in \ddot{\mathbb{S}}$. The couple (X, Y) is a dyad if and only if $aX \cup Y$ is a dendrite.

Intuitively, this theorem asserts that, if (X, Y) is a dyad, then we cancel out all cycles of Y (*i.e.*, we obtain an acyclic complex), whenever we cancel out those of X (by the way of a cone). Furthermore, it asserts that, if we are able to cancel all cycles of Y by such a way, then (X, Y) is a dyad.

Relative dendrites: the cancelation theorem

A couple (X, Y) which is a dyad, and a cone aX:



Theorem: Let $(X, Y) \in \ddot{\mathbb{S}}$. The couple (X, Y) is a dyad if and only if $aX \cup Y$ is a dendrite.

Intuitively, this theorem asserts that, if (X, Y) is a dyad, then we cancel out all cycles of Y (*i.e.*, we obtain an acyclic complex), whenever we cancel out those of X (by the way of a cone). Furthermore, it asserts that, if we are able to cancel all cycles of Y by such a way, then (X, Y) is a dyad.

Homotopic pairs

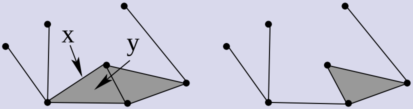
Motivation:

We want to make a link between previous completions and (simple) homotopy.

Homotopic pairs: simple homotopy

Let $X \in \mathbb{S}$. A face $x \in X$ is free for X if x is a proper face of exactly one face y of X, such a pair (x, y) is a free pair for X.
If (x, y) is a free pair for X, $Y = X \setminus \{x, y\}$ is an elementary collapse of X and X is an elementary expansion of Y.
The complex X collapses onto Y if Y may be obtained from X by elementary collapses.
Two complexes are (simply) homotopic if one of them may be obtained from the other by elementary collapses and expansions.

An object X and an elementary collapse of X:



Homotopic pairs: contractible complexes

A complex is (simply) contractible if it is simply homotopic to a single vertex.

Proposition: Any contractible complex is a dendrite.

Homotopic pairs: contractible complexes

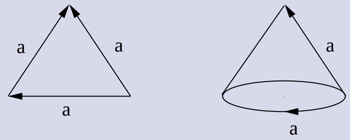
A complex is (simply) contractible if it is simply homotopic to a single vertex.

Proposition: Any contractible complex is a dendrite.

Homotopic pairs: the dunce hat

Proposition: Any contractible complex is a dendrite.

The dunce hat is a contractible complex. It is an example of an object that is a dendrite but not a ramification.



Homotopic pairs: homology and homotopy

Proposition: Any contractible complex is a dendrite.

There exist some dendrites that are not contractible. The punctured Poincaré homology sphere provides an example of this fact. Thus, the collection of contractible complexes is a proper subset of the collection of dendrites.

We would like to have a better understanding of the links between dendrites (or dyads) and homotopy.

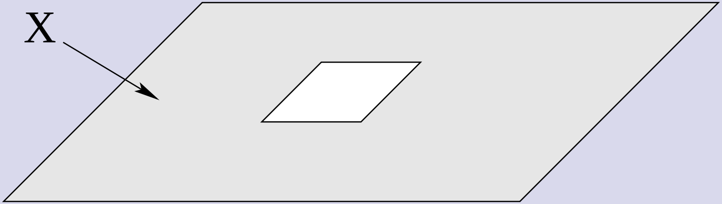
Homotopic pairs: homology and homotopy

Proposition: Any contractible complex is a dendrite.

There exist some dendrites that are not contractible. The punctured Poincaré homology sphere provides an example of this fact. Thus, the collection of contractible complexes is a proper subset of the collection of dendrites.

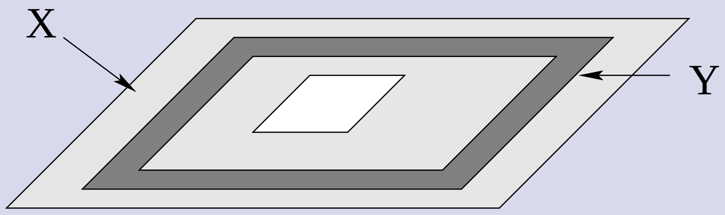
We would like to have a better understanding of the links between dendrites (or dyads) and homotopy.

Homotopic pairs: a basic idea



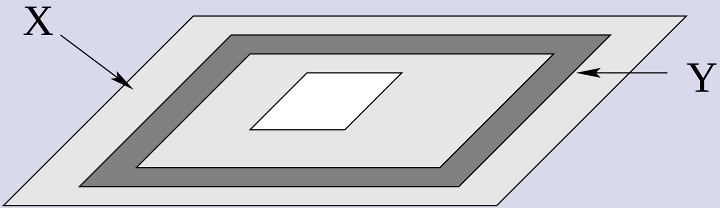
We try to take benefit of all the previous completions that act on pairs rather than a single complex.

Homotopic pairs: a basic idea



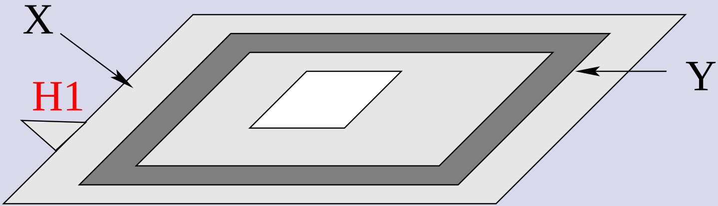
The pair (Y, X) is a dyad:
we observe that, in this example, it is possible to continuously deform X onto Y while keeping Y inside X.

Homotopic pairs: a basic idea



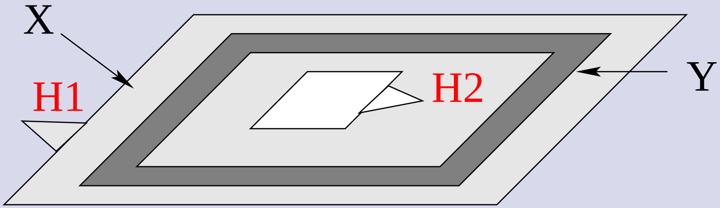
It follows the idea to consider 4 elementary moves H1,H2,H3,H4.
This moves may be seen as “relative collapses/expansions”.

Homotopic pairs: a basic idea



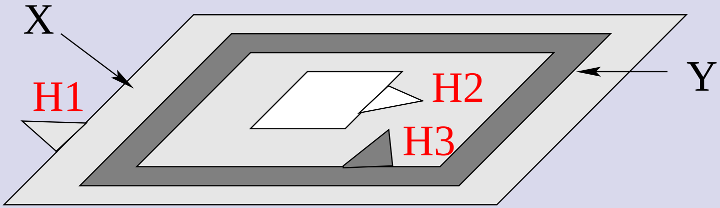
H1 is an expansion of X.

Homotopic pairs: a basic idea



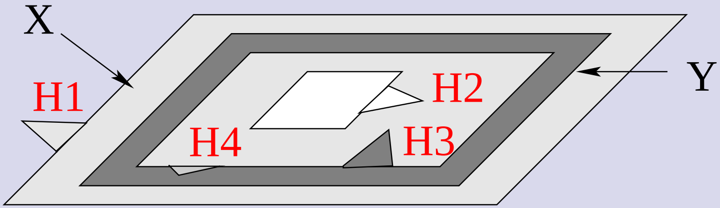
H2 is a collapse of X constrained to keep Y inside X.

Homotopic pairs: a basic idea



H3 is an expansion of Y constrained to keep Y inside X.

Homotopic pairs: a basic idea



H4 is a collapse of Y.

Homotopic pairs: the axioms

If $X, Y \in \mathbb{S}$, we write $X \stackrel{E}{\longmapsto} Y$, whenever Y is an elementary expansion of X. We define four completions on $\ddot{\mathbb{S}}$:

For any (R, S), (R, T), (S, T) in $\ddot{\mathbb{S}}$,

-> If $(R, S) \in \ddot{\mathcal{K}}$ and $S \stackrel{E}{\longmapsto} T$, then $(R, T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{H}}1 \rangle$

-> If $(R, T) \in \ddot{\mathcal{K}}$ and $S \stackrel{E}{\longmapsto} T$, then $(R, S) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{H}}2 \rangle$

-> If $(R, T) \in \ddot{\mathcal{K}}$ and $R \stackrel{E}{\longmapsto} S$, then $(S, T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{H}}3 \rangle$

-> If $(S, T) \in \ddot{\mathcal{K}}$ and $R \stackrel{E}{\longmapsto} S$, then $(R, T) \in \ddot{\mathcal{K}}$. $\langle \ddot{\mathrm{H}}4 \rangle$

We set $\ddot{\mathbb{I}} = \{(X, X) \mid X \in \mathbb{S}\}$ and $\ddot{\mathbb{H}} = \langle \ddot{\mathbb{I}}; \ddot{\mathrm{H}}1, \ddot{\mathrm{H}}2, \ddot{\mathrm{H}}3, \ddot{\mathrm{H}}4 \rangle$.

Each element of $\ddot{\mathbb{H}}$ is a **homotopic pair**.

Proposition:
We have $\ddot{\mathbb{H}} \subseteq \ddot{\mathbb{X}}$, *i.e.*, any homotopic pair is a dyad.
Thus, we have $\ddot{\mathbb{H}} \subseteq \ddot{\mathbb{X}} = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{Y}}2, \ddot{\mathrm{T}}, \ddot{\mathrm{U}}, \ddot{\mathrm{L}} \rangle$.

Theorem: We have $\ddot{\mathbb{H}} = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{T}}, \ddot{\mathrm{U}}, \ddot{\mathrm{L}} \rangle$.

Homotopic pairs: The homotopic pairs theorem

Proposition:
We have $\ddot{\mathbb{H}} \subseteq \ddot{\mathbb{X}}$, *i.e.*, any homotopic pair is a dyad.
Thus, we have $\ddot{\mathbb{H}} \subseteq \ddot{\mathbb{X}} = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{Y}}2, \ddot{\mathrm{T}}, \ddot{\mathrm{U}}, \ddot{\mathrm{L}} \rangle$.

Theorem: We have $\ddot{\mathbb{H}} = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{T}}, \ddot{\mathrm{U}}, \ddot{\mathrm{L}} \rangle$.

Homotopic pairs: The homotopic pairs theorem

Proposition:
We have $\ddot{\mathbb{H}} \subseteq \ddot{\mathbb{X}}$, *i.e.*, any homotopic pair is a dyad.
Thus, we have $\ddot{\mathbb{H}} \subseteq \ddot{\mathbb{X}} = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{Y}}2, \ddot{\mathrm{T}}, \ddot{\mathrm{U}}, \ddot{\mathrm{L}} \rangle$.

Theorem: We have $\ddot{\mathbb{H}} = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{T}}, \ddot{\mathrm{U}}, \ddot{\mathrm{L}} \rangle$.

- In a single framework we can express some notions linked to homology and homotopy.
- We obtain global properties for simple homotopy.
- The collection of all homotopic pairs is fully characterized by these properties.
- The distinction between dyads ($\simeq$ homology) and homotopy is clear.

Proposition:
We have $\ddot{\mathbb{H}} \subseteq \ddot{\mathbb{X}}$, *i.e.*, any homotopic pair is a dyad.
Thus, we have $\ddot{\mathbb{H}} \subseteq \ddot{\mathbb{X}} = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{Y}}2, \ddot{\mathrm{T}}, \ddot{\mathrm{U}}, \ddot{\mathrm{L}} \rangle$.

Theorem: We have $\ddot{\mathbb{H}} = \langle \ddot{\mathbb{C}}; \ddot{\mathrm{Y}}1, \ddot{\mathrm{T}}, \ddot{\mathrm{U}}, \ddot{\mathrm{L}} \rangle$.

- In a single framework we can express some notions linked to homology and homotopy.
- We obtain global properties for simple homotopy.
- The collection of all homotopic pairs is fully characterized by these properties.
- The distinction between dyads ($\simeq$ homology) and homotopy is clear.

Conclusion

Conclusion

- We introduced completions as a language for constructive and dynamic descriptions of collections of complexes.
- These completions correspond to global topological properties of these collections.
- We introduced the notions of dendrites, dyads, relative dendrites, confluence, homotopic pairs, and we gave several theorems which show the deep links between these collections.

Thank you for your attention !

Dendrites: duality

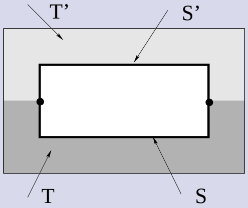
Let $A \in \mathbb{C}$ and $X \preceq A$. The dual of X for A is the simplicial complex X_A^*, such that $X_A^* = \{\underline{A} \setminus x \mid x \in A \setminus X\}$, where $\underline{A}$ is the ground set of A, *i.e.*, $\underline{A} = \cup\{x \in X \mid dim(x) = 0\}$.

Dendrites: duality

Let $A \in \mathbb{C}$ and $X \preceq A$. The dual of X for A is the simplicial complex X^*_A, such that $X^*_A = \{\underline{A} \setminus x \mid x \in A \setminus X\}$, where $\underline{A}$ is the ground set of A, *i.e.*, $\underline{A} = \cup\{x \in X \mid dim(x) = 0\}$.

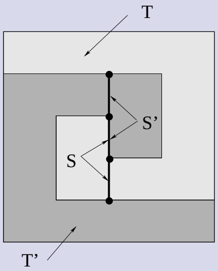
Proposition: Let $A \in \mathbb{C}$ and $X \preceq A$. The complex X is a dendrite if and only if X^*_A is a dendrite.

Relative dendrites: example (1)



In this figure $(S \cup S', T \cup S')$ and $(T \cup S', T \cup T')$ are dyads. Then, using $\langle \ddot{\mathrm{T}} \rangle$, it is possible to generate directly $(S \cup S', T \cup T')$ without the completions defining relative dendrites.

Relative dendrites: example (2)

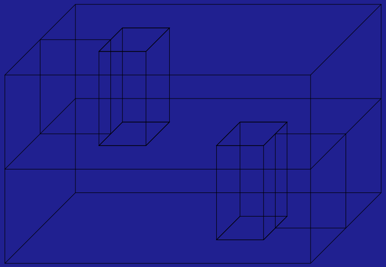


Now, in this figure, we observe that we can generate $(S \cup S', T \cup T')$ with the completions for relative dendrites. Nevertheless $(S \cup S', T \cup S')$ is not a dyad ($S \cup S'$ is acyclic, while $T \cup S'$ is not). Thus, it is not possible to generate, in a straightforward manner, the relative dendrite $(S \cup S', T \cup T')$ with the previous completions for dyads.

Homotopic pairs: collapsible complexes

Let X be a triangulation of a square. If X collapses onto Y, then Y collapses onto a single vertex.

This property is not true if we consider a cube!

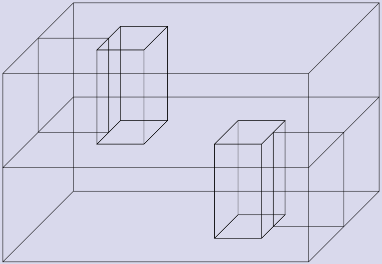


The Bing's house

Homotopic pairs: collapsible complexes

Let X be a triangulation of a square. If X collapses onto Y, then Y collapses onto a single vertex.

This property is not true if we consider a cube!



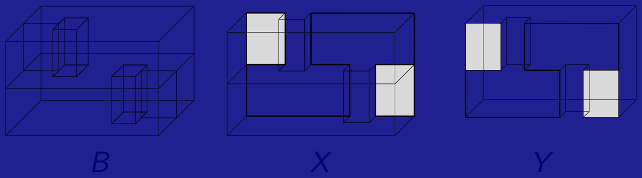
The Bing's house

Homotopic pairs: the Bing's house

Proposition: Any contractible complex is a dendrite.

For example, the Bing's house is a dendrite.

We have $B = X \cup Y$, and X, Y, and $X \cap Y$ are dendrites.



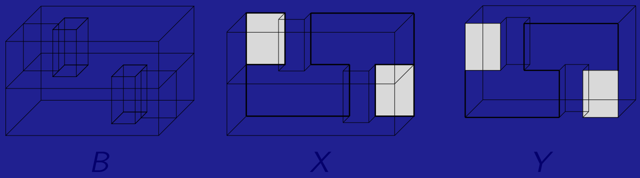
In fact the Bing's house is a ramification.

Homotopic pairs: the Bing's house

Proposition: Any contractible complex is a dendrite.

For example, the Bing's house is a dendrite.

We have $B = X \cup Y$, and X, Y, and $X \cap Y$ are dendrites.



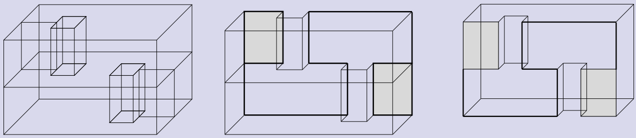
In fact the Bing's house is a ramification.

Homotopic pairs: the Bing's house

Proposition: Any contractible complex is a dendrite.

For example, the Bing's house is a dendrite.

We have $B = X \cup Y$, and X, Y, and $X \cap Y$ are dendrites.



B X Y

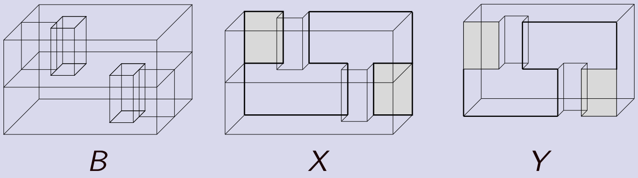
In fact the Bing's house is a ramification.

Homotopic pairs: the Bing's house

Proposition: Any contractible complex is a dendrite.

For example, the Bing's house is a dendrite.

We have $B = X \cup Y$, and X, Y, and $X \cap Y$ are dendrites.



In fact the Bing's house is a ramification.

Homotopic pairs: simple homotopy

Theorem:
Let $X, Y \in \mathbb{S}$ and let λY be a copy of Y disjoint from X. The complexes X and Y are simply homotopic if and only if there exists $K \in \mathbb{S}$ such that K collapses onto both X and λY.

Theorem (Whitehead):
Let $X, Y \in \mathbb{S}$. The complexes X and Y are simply homotopic if and only if there exists $K \in \mathbb{S}$ and there exists a stellar sub-division $\widetilde{Y}$ of Y, such that K collapses onto both X and $\widetilde{Y}$.

Homotopic pairs: simple homotopy

Theorem:
Let $X, Y \in \mathbb{S}$ and let λY be a copy of Y disjoint from X. The complexes X and Y are simply homotopic if and only if there exists $K \in \mathbb{S}$ such that K collapses onto both X and λY.

Theorem (Whitehead):
Let $X, Y \in \mathbb{S}$. The complexes X and Y are simply homotopic if and only if there exists $K \in \mathbb{S}$ and there exists a stellar sub-division $\widetilde{Y}$ of Y, such that K collapses onto both X and $\widetilde{Y}$.