Mathematical Morphology on a Few Discrete Structures

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Lattices and information processing

Lattices: core mathematical structure in many information processing problems.

Examples:

- soft computing (fuzzy sets, bipolar information),
- knowledge representation,
- logics,
- formal concept analysis,
- automated reasoning,
- decision making,
- image processing and understanding,
- information retrieval,
- etc.

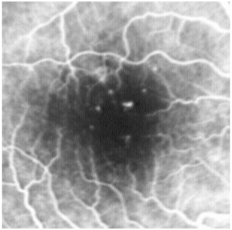
Mathematical morphology on complete lattices.

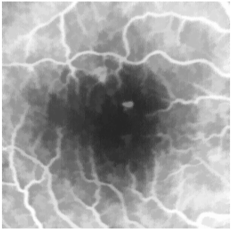
Mathematical Morphology for Spatial Information

Matheron (mid-1960's), Serra (1982)

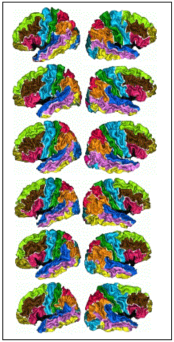
- A theory of space.
- Widely used in image processing and interpretation.
- At different levels (local, regional, structural...).
- For different tasks (filtering, enhancement, segmentation, interpretation, spatial knowledge modeling...).

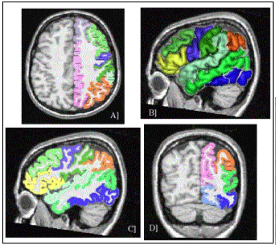
Filtering

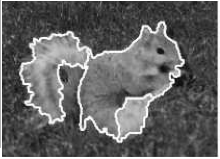


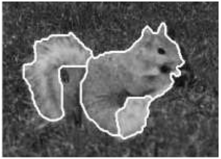


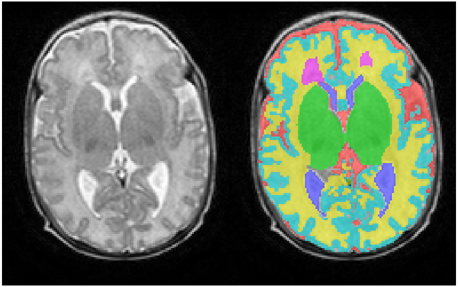
Segmentation and interpretation



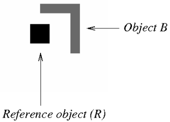


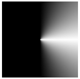


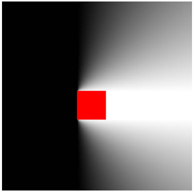




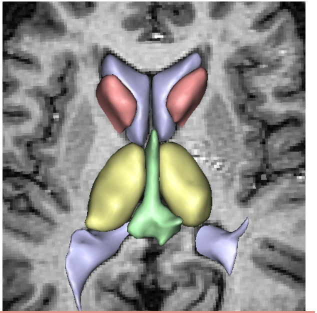
Knowledge modeling What is the region *to the right of R*? Is *B to the right of R (and to which degree)*?



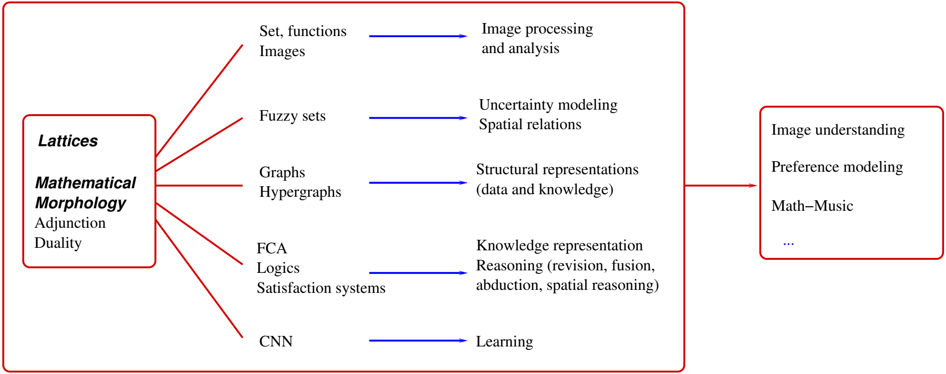




Spatial reasoning



Symbolic and structural representations and reasoning



Formal framework: complete lattices

- Lattice: $(\mathcal{T}, \leq)$ ($\leq$ partial ordering) such that $\forall (x, y) \in \mathcal{T}, \exists x \vee y$ and $\exists x \wedge y$.
- Complete lattice: every family of elements (finite or not) has a smallest upper bound and a largest lower bound.
- Examples of complete lattices:
  - $(\mathcal{P}(E), \subseteq)$: complete lattice, Boolean (complemented and distributive)
  - functions of $\mathbb{R}^n$ in $\overline{\mathbb{R}}$ for the partial ordering $\leq$: $f \leq g \Leftrightarrow \forall x \in \mathbb{R}^n, \;\; f(x) \leq g(x)$
  - partitions
  - fuzzy sets, bipolar fuzzy sets
  - rough sets and fuzzy rough sets
  - graphs, hypergraphs
  - logics (propositional logics, modal logics...)
  - formal concepts
  - ...

Mathematical morphology in a nutshell

Dilation: operation in complete lattices that commutes with the supremum.
Erosion: operation in complete lattices that commutes with the infimum.

$\Rightarrow$ applies in any mathematical framework endowed with a lattice structure.

Using a structuring element:

- dilation as a degree of conjunction: $\delta_B(X) = \{x \in \mathcal{S} \mid B_x \cap X \neq \emptyset\}$,
- erosion as a degree of implication: $\varepsilon_B(X) = \{x \in \mathcal{S} \mid B_x \subseteq X\}$.

Derived operators: opening, closing, conditional (geodesic) operations, gradient...
Relaxing the assumption on invariance under translation: structuring elements varying in space (ex: projective geometry, omnidirectional images...).

Adjunctions

$\delta : (\mathcal{T}, \leq) \to (\mathcal{T}', \leq')$, $\varepsilon : (\mathcal{T}', \leq') \to (\mathcal{T}, \leq)$, (ε, δ) adjunction if:

$$\forall x \in \mathcal{T}, \forall y \in \mathcal{T}',\ \delta(x) \leq' y \Leftrightarrow x \leq \varepsilon(y)$$

Properties:

- $\delta(0) = 0'$ and $\varepsilon(I') = I$.
- (ε, δ) adjunction $\Rightarrow \varepsilon =$ algebraic erosion and $\delta =$ algebraic dilation.
- δ increasing $=$ algebraic dilation iff $\exists \varepsilon$ such that (ε, δ) is an adjunction $\Rightarrow \varepsilon =$ algebraic erosion and $\varepsilon(x) = \bigvee\{y \in \mathcal{T},\ \delta(y) \leq' x\}$.
- $\varepsilon\delta \geq Id$ and $\delta\varepsilon \leq Id'$.
- $\varepsilon\delta\varepsilon = \varepsilon$ and $\delta\varepsilon\delta = \delta$; $\varepsilon\delta\varepsilon\delta = \varepsilon\delta$ and $\delta\varepsilon\delta\varepsilon = \delta\varepsilon$.
- δ and ε increasing such that $\delta\varepsilon \leq Id'$ and $\varepsilon\delta \geq Id \Rightarrow (\varepsilon, \delta)$ adjunction.

- Algebraic opening: γ increasing, idempotent and anti-extensive.
- Algebraic closing: φ increasing, idempotent and extensive.
- Examples: $\gamma = \delta\varepsilon$ and $\varphi = \varepsilon\delta$ with (ε, δ) adjunction.

Lattice of fuzzy sets and fuzzy morphology

- Space $\mathcal{S}$ (e.g. $\mathbb{Z}^n$ or $\mathbb{R}^n$)
- $\mathcal{F}$: set of fuzzy sets on $\mathcal{S}$ – $\mu \in \mathcal{F}$, $\mu : \mathcal{S} \to [0, 1]$.
- Partial ordering: $\forall(\mu_1, \mu_2) \in \mathcal{F}^2, \mu_1 \leq \mu_2 \Leftrightarrow \forall x \in \mathcal{S}, \mu_1(x) \leq \mu_2(x)$
- $(\mathcal{F}, \leq)$ = complete lattice
- $\wedge = \min$
- $\vee = \max$
- Algebraic dilation and erosion: as in any complete lattice
- Residuated lattice $(\mathcal{F}, \leq, t, I)$, t = t-norm (fuzzy conjunction), I = fuzzy implication.

Fuzzy dilation and erosion of μ by ν:

$$\delta_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} t(\nu(x - y), \mu(y)) \quad \varepsilon_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} I(\nu(y - x), \mu(y))$$

Properties: as in classical morphology.

Structural information: spatial relations

Expression of several spatial relations in terms of morphological operators:

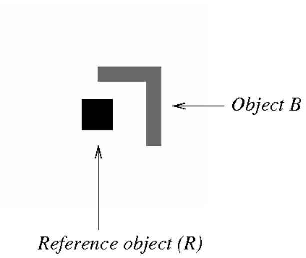
- adjacency
- distance (nearest point distance, Hausdorff distance)
- relative direction
- more complex relations (between, along, parallel, crossing...)

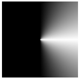
Two classes of relations:

- well defined in the crisp case
- vague even if objects are well defined

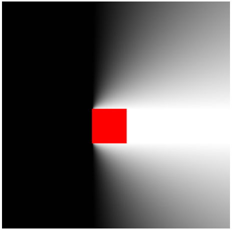
Example of directional relation

Filling the semantic gap: fuzzy representation of concepts in concrete domains.

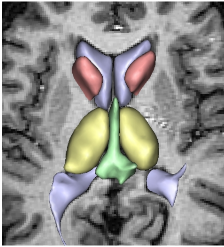




ν_{right}



$\delta_{\nu_{right}}(R)$



$$Right(B, R) = f(\delta_{\nu_{right}}(R)(x), x \in B)$$

Extends directly to fuzzy objects.

$\Rightarrow$ use in spatial reasoning, for knowledge-based object segmentation and recognition.

Hypergraphs

- Underlying space: $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V}$ the set of vertices and $\mathcal{E}$ the set of hyperedges.
- Hypergraph: $H = (V, E)$ with $V \subseteq \mathcal{V}$ and $E \subseteq \mathcal{E}$, $E = ((e_i)_{i \in I})$ $(e_i \subseteq V)$.
- $v(e)$ = set of vertices forming the hyperedge e (equivalence).
- Isolated vertex: $x \in V \setminus \cup_{i \in I} v(e_i) = V_{\setminus E}$.
- Dual hypergraph: $H^* = (V^* \simeq E,\ E^* \simeq (H(x))_{x \in V})$ with $H(x) = \text{star}(x) = \{e \mid x \in e\}$.

Joint work with Alain Bretto (CVIU 2013, DAM 2015, DGCI 2019)

Partial ordering and complete lattices

- On vertices: $\mathcal{T}_1 = (\mathcal{P}(\mathcal{V}), \subseteq)$.
- On hyperedges: $\mathcal{T}_2 = (\mathcal{P}(\mathcal{E}), \subseteq)$.
- On $\mathcal{H}$: $\mathcal{T}_3 = (\{H\}, \preceq)$, with $\{H = (V, E)\}$ = set of hypergraphs defined on $(\mathcal{V}, \mathcal{E})$ such that $\forall e \in E, v(e) \subseteq V$.
  - Partial ordering: $\forall (H_1 = (V_1, E_1), H_2 = (V_2, E_2)) \in \mathcal{T}_3^2$,

$$H_1 \preceq H_2 \Leftrightarrow V_1 \subseteq V_2 \textit{ and } E_1 \subseteq E_2$$

  - Inf and Sup:

$$H_1 \wedge H_2 = (V_1 \cap V_2, E_1 \cap E_2)$$

$$H_1 \vee H_2 = (V_1 \cup V_2, E_1 \cup E_2)$$

    and extensions to any family.
  - Smallest element: $H_\emptyset = (\emptyset, \emptyset)$, largest element: $\mathcal{H} = (\mathcal{V}, \mathcal{E})$.

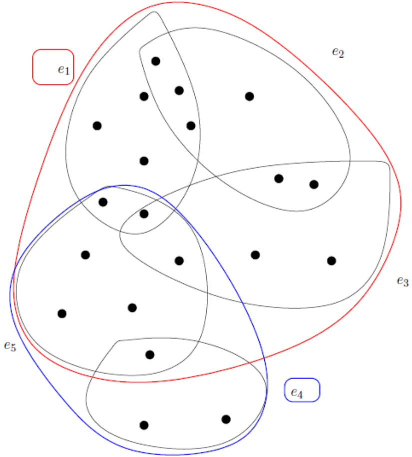
Algebraic erosions and dilations as in any complete lattices.

Structuring element: binary relation between two elements.

Example: $\delta : (\mathcal{P}(\mathcal{E}), \subseteq) \rightarrow (\mathcal{P}(\mathcal{V}), \subseteq)$

$$\forall e \in E, B_e = \delta(\{e\}) = \{x \in \mathcal{V} \mid \exists e' \in \mathcal{E}, x \in e' \text{ and } v(e) \cap v(e') \neq \emptyset\}$$

$$= \cup\{v(e') \mid v(e') \cap v(e) \neq \emptyset\}$$



Morpho-Logics

Joint work with Jérôme Lang, IRIT (now at LAMSADE): mathematical morphology on logical formulas, via the models, in propositional logic [IPMU-2000, TCIS-2002, AI].

- Set of all models of a formula φ: $[\![\varphi]\!] = \{\omega \in \Omega \mid \omega \models \varphi\}$
- Lattice structure on the set of all models $\Leftrightarrow$ lattice structure on the set of formulas (up to an equivalence relation).
- $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$,
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$,
- $[\![\varphi]\!] \subseteq [\![\psi]\!]$ iff $\varphi \models \psi$, and φ is consistent iff $[\![\varphi]\!] \neq \emptyset$
- Algebraic dilations and erosions as in any complete lattice.

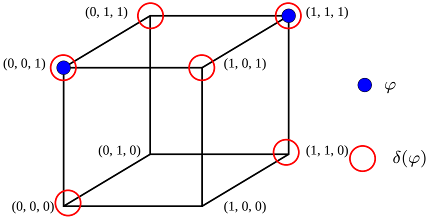
Morphological dilation of a formula φ with a structuring element B:

$$[\![\delta_B(\varphi)]\!] = \delta_B([\![\varphi]\!]) = \{\omega \in \Omega \mid \check{B}_\omega \wedge \varphi \textit{ consistent}\}.$$

Morphological erosion:

$$[\![\varepsilon_B(\varphi)]\!] = \varepsilon_B([\![\varphi]\!]) = \{\omega \in \Omega \mid B_\omega \models \varphi\}.$$

B: a relation between worlds, e.g. neighborhood, distance.



Example of a dilation of size 1 (Hamming distance):
$\varphi = (a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge c)$ and $\delta(\varphi) = (\neg a \vee b \vee c) \wedge (a \vee \neg b \vee c)$.

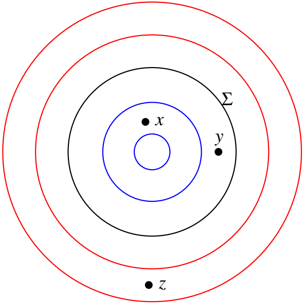
Reasoning with morpho-logics (in PL)

Joint work with Ramón Pino Pérez and Carlos Uzcátegui, Los Andes, Merida, Venezuela (now in Ecuador and Columbia) [ECSQARU-2001, KR-2004, ECAI-2006, AI].

- Revision.
- Merging (fusion).
- Abductive reasoning.
- Mediation.

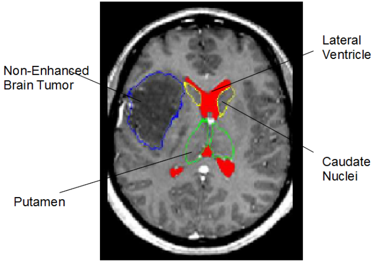
Using dilations and erosions

Morphological partial ordering: stratification of the models from successive dilations and erosions.



Example: $x \preceq_f y \preceq_f z$

Abductive reasoning: example in image understanding



Pathological brain with a tumor

$$\mathcal{K} \models (\gamma \to \mathcal{O})$$

Compute the "best" explanation to the observations taking into account the expert knowledge (e.g. formalized in description logic).

Other logics

Modal logics [JANCL-2002]

Accessibility relation	Structuring element
$\square$	ε
$\diamond$	δ

Description logics (joint work with Jamal Atif, LAMSADE, and Céline Hudelot, MICS) [FSS-2008, IEEE SMC-2014]: δ and ε as binary predicates, ontological reasoning.

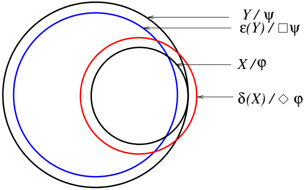
Satisfaction systems and institutions (joint work with Marc Aiguier, MICS) [AI-2018, IJAR-2018, JANCL]:

- General framework for many logics.
- Revision based on relaxations.
- Abduction based on cuttings and retractions.
- Dual operators from dilations and erosions.
- Towards spatial reasoning.

Spatial reasoning using modal morpho-logic

Examples in mereotopology (with $\Box \equiv \varepsilon$ and $\Diamond \equiv \delta$):

- tangential part: $\varphi \to \psi$ and $\Diamond\varphi \wedge \neg\psi$ consistent, or $\varphi \to \psi$ and $\varphi \wedge \neg\Box\psi$ consistent



- non tangential part: $\Diamond\varphi \to \psi$, or $\varphi \to \Box\psi$
- external connection (adjacency): $\varphi \wedge \psi$ inconsistent and $\Diamond\varphi \wedge \psi$ consistent (or $\varphi \wedge \Diamond\psi$ consistent)

Formal Concept Analysis (FCA) (Ganter et al. 1997)

- Set of objects G.
- Set of attributes M.
- Relation $I \subseteq G \times M$: $(g, m) \in I$ = object g has attribute m.
- Formal context: $\mathbb{K} = (G, M, I)$.
- Derivation operators:

$$\forall X \subseteq G, \alpha(X) = \{m \in M \mid \forall g \in X, (g, m) \in I\}$$

$$\forall Y \subseteq M, \beta(Y) = \{g \in G \mid \forall m \in Y, (g, m) \in I\}$$

$\mathbb{K}$	even	odd	prime	square
1		$\times$		$\times$
2	$\times$		$\times$	
3		$\times$	$\times$	

$$\alpha(\{2, 3\}) = \{p\}$$
$$\beta(\{o, p\}) = \{3\}$$

I as a structuring element

Galois connection

- (α, β) is a Galois connection between the posets $(\mathcal{P}(G), \subseteq)$ and $(\mathcal{P}(M), \subseteq)$:

$$\forall X \in \mathcal{P}(G), \forall Y \in \mathcal{P}(M), Y \subseteq \alpha(X) \Leftrightarrow X \subseteq \beta(Y)$$

- (X, Y) is a formal concept $\Leftrightarrow \alpha(X) = Y$ and $\beta(Y) = X$
  Formal concept $a = (e(a), i(a))$, extent $e(a) \subseteq G$, intent $i(a) \subseteq M$.
  $\Rightarrow$ complete lattice $(\mathbb{C}, \preceq)$.
- Partial ordering: $(X_1, Y_1) \preceq (X_2, Y_2) \Leftrightarrow X_1 \subseteq X_2$ $(\Leftrightarrow Y_2 \subseteq Y_1)$.
- Smallest element: $(\emptyset, M)$. Largest element: $(G, \emptyset)$.
- Infimum and supremum:

$$\bigwedge_{t \in T} (X_t, Y_t) = \left(\bigcap_{t \in T} X_t, \alpha\big(\beta(\bigcup_{t \in T} Y_t)\big) \right),$$

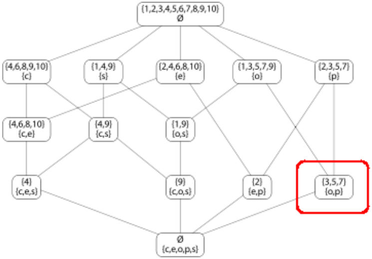
$$\bigvee_{t \in T} (X_t, Y_t) = \left(\beta\big(\alpha(\bigcup_{t \in T} X_t)\big), \bigcap_{t \in T} Y_t \right).$$

A simple example

$\mathbb{K}$	composite	even	odd	prime	square
1			×		×
2		×		×	
3			×	×	
4	×	×			×
5			×	×	
6	×	×			
7			×	×	
8	×	×			
9	×		×		×
10	×	×			

$\mathbb{K} = (G = \{1, 2 \ldots 10\}, M = \{c, e, o, p, s\}, I)$

composite = integer $>$ 1 and non prime



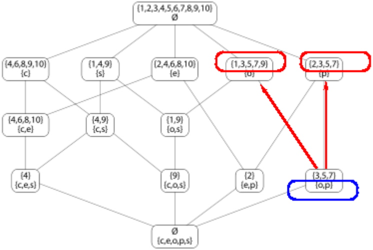
FCA: Adjunction and Galois connection

Equivalent concepts by reversing the order on one space. (CLA 2011)

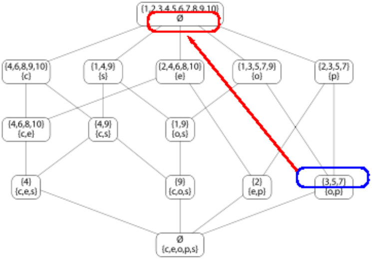
$\delta : A \rightarrow B$, $\varepsilon : B \rightarrow A$	$\alpha : B \rightarrow A, \beta : A \rightarrow B$
$\delta(a) \leq_B b \Leftrightarrow a \leq_A \varepsilon(b)$	$a \leq_A \alpha(b) \Leftrightarrow b \leq_B \beta(a)$
	($\Leftrightarrow \beta(a) \leq'_B b$ with $\leq'_B \equiv \geq_B$)
increasing operators	decreasing operators
$\varepsilon\delta\varepsilon = \varepsilon, \delta\varepsilon\delta = \delta$	$\alpha\beta\alpha = \alpha, \beta\alpha\beta = \beta$
$\varepsilon\delta$ = closing, $\delta\varepsilon$ = opening	$\alpha\beta$ and $\beta\alpha$ = closings
$Inv(\varepsilon\delta) = \varepsilon(B)$, $Inv(\delta\varepsilon) = \delta(A)$	$Inv(\alpha\beta) = \alpha(B)$, $Inv(\beta\alpha) = \beta(A)$
$\varepsilon(B)$ = Moore family	$\alpha(B)$ and $\beta(A)$ = Moore families
$\delta(A)$ = dual Moore family	
δ = dilation: $\delta(\vee_A a_i) = \vee_B(\delta(a_i))$	$\alpha(\vee_B b_i) = \wedge_A \alpha(b_i)$
ε = erosion: $\varepsilon(\wedge_B b_i) = \wedge_A(\varepsilon(b_i))$	$\beta(\vee_A a_i) = \wedge_B \beta(a_i)$ (anti-dilation)

($M \subseteq \mathcal{L}$ is a Moore family if any element of $\mathcal{L}$ has a smallest upper bound in M)

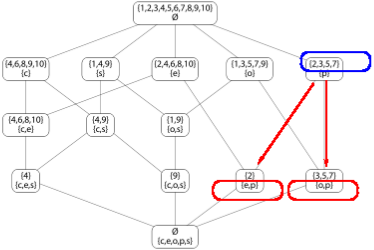
$\delta_I(\{o, p\}) = \{1, 2, 3, 5, 7, 9\}$



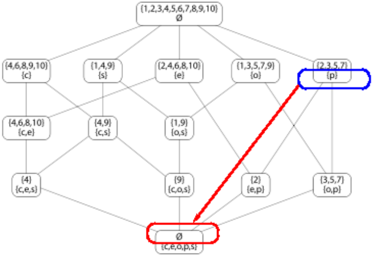
$\varepsilon_I(\{3, 5, 7\}) = \emptyset$



$\delta_I^*(\{2, 3, 5, 7\}) = \{e, p, o\}$



$\varepsilon_I^*(\{p\}) = \emptyset$



Mathematical operators over concept lattices: two approaches

1. Based on the notion of structuring element, defined as a ball of radius 1 of some distance function on G derived from a distance on $\mathbb{C}$.
2. Directly from a distance on $\mathbb{C}$.

Joint work with Jamal Atif, Céline Hudelot, Felix Distel (ICFCA 2013, IJUFKS 2016).

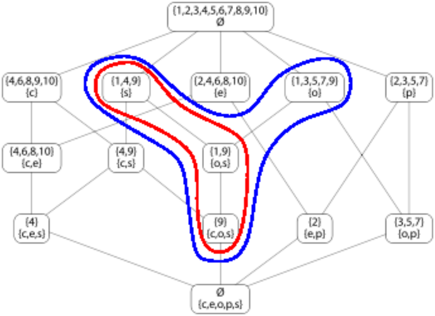
Examples

$$a = (\{1,9\}, \{o,s\})$$

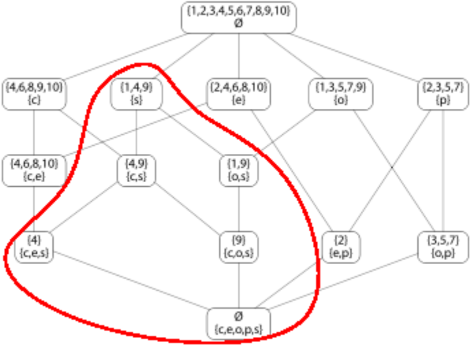
$$a_1 = (\{1,4,9\}, \{s\}) \qquad a_2 = (\{1,3,5,7,9\}, \{o\}) \qquad a_3 = (\{9\}, \{c,o,s\})$$

$$d\omega_G(a,a_1) = d\omega_G(a,a_3) = 1 \Rightarrow \delta^1_G(\{a\}) = \{a, a_1, a_3\}$$

$$d\omega_M(a,a_1) = d\omega_M(a,a_2) = d\omega_M(a,a_3) = 1 \Rightarrow \delta^1_M(\{a\}) = \{a, a_1, a_2, a_3\}$$



Dilation of $\{a_1\} = \{(\{1,4,9\},\{s\})\}$ using as a structuring element a ball of d_{ω_G} for each irreducible element of its decomposition $(\{4\},\{c,e,s\}) \vee (\{1,9\},\{o,s\}) \vee (\{9\},\{c,o,s\})$:



$\neq \delta_G(\{a_1\})$!

Links with other lattices (hypergraphs...) and extensions (fuzzy sets, rough sets, F-transforms...). (LFA 2015, IJUFKS 2016)

Lattices

$(\mathcal{P}(G), \subseteq)$
$(\mathcal{P}(M), \subseteq)$

$(L^G, \preceq_F, \wedge^F, \vee^F, *, \rightarrow)$
$(L^M, \preceq_F, \wedge^F, \vee^F, *, \rightarrow)$

$(\mathbb{C}^F, \preceq_{FC}, \wedge^{FC}, \vee^{FC})$

$(\delta, \varepsilon) \longleftrightarrow \alpha = I^\Delta, \beta = I^\Delta_M$

$I \longleftrightarrow$ Structuring element / Relation in approximation spaces

Adjunctions

$$\begin{cases} \delta_I = I^\Pi_M = \overline{R}^M = f^\downarrow : & \mathcal{P}(M) \rightarrow \mathcal{P}(G) \\ \varepsilon_I = I^N = \underline{R}^G = F^\downarrow : & \mathcal{P}(G) \rightarrow \mathcal{P}(M) \end{cases}$$

$$\begin{cases} \delta^*_I = I^\Pi = \overline{R}^G = F^\uparrow : & \mathcal{P}(G) \rightarrow \mathcal{P}(M) \\ \varepsilon^*_I = I^N_M = \underline{R}^M = f^\uparrow : & \mathcal{P}(M) \rightarrow \mathcal{P}(G) \end{cases}$$

Valuations
...

→ metrics d →

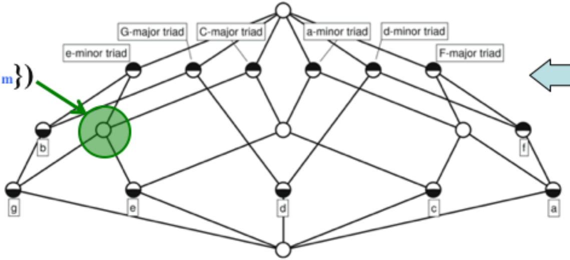
$\delta : \mathbb{C}^F \rightarrow \mathbb{C}^F$
$\varepsilon : \mathbb{C}^F \rightarrow \mathbb{C}^F$

A concept lattice for the diatonic scale

	C-major triad	d-minor triad	e-minor triad	F-major triad	G-major triad	a-minor triad
c	X			X		X
d		X			X	
e	X		X			X
f		X		X		
g	X		X		X	
a		X		X		X
b			X		X	

	C_M	E_m				
mi	X	X				
sol	X	X				

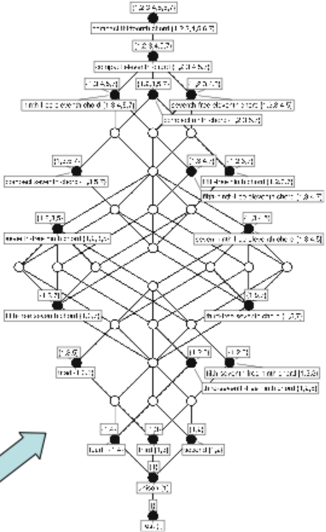
({*mi*, *sol*},{C_M, E_m})



[R. Wille & R. Wille-Henning, « Towards a Semantology of Music », ICCS 2007, Springer, 2007]

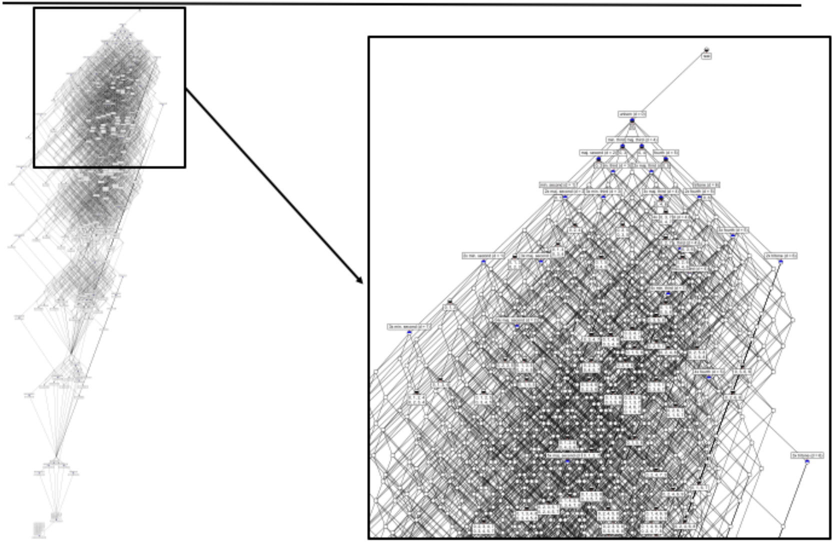
A different concept lattice for the diatonic scale

	{}	{1}	{1,2}	{1,3}	{1,4}	{1,2,3}	{1,3,7}	{1,2,5}	{1,5,7}	{1,3,5}	{1,3,5,7}	{1,2,3,5}	{1,3,4,7}	{1,3,4,5}	{1,2,3,7}	{1,3,4,5,7}	{1,2,3,5,7}	{1,2,3,4,5}	{1,2,3,4,5,7}	{1,2,3,4,5,6,7}
rest { }	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
unison {1}		X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
second {1,2}			X			X	X	X	X		X	X	X	X	X	X	X	X	X	X
third {1,3}				X		X	X		X	X	X	X	X	X	X	X	X	X	X	X
fourth {1,4}					X		X	X	X	X	X	X	X	X	X	X	X	X	X	X
fifth-seventh-free ninth chord {1,2,3}						X						X		X	X	X	X	X	X	X
fifth-free seventh chord {1,3,7}							X				X	X	X		X	X	X	X	X	X
third-seventh-free ninth chord {1,2,5}								X				X	X	X		X	X	X	X	X
third-free seventh chord {1,5,7}									X		X		X	X	X	X	X	X	X	X
triad {1,3,5}										X	X	X		X		X	X	X	X	X
compact seventh chord {1,3,5,7}											X					X	X		X	X
seventh-free ninth chord {1,2,3,5}												X				X	X	X	X	X
fifth-ninth-free eleventh chord {1,3,4,7}													X			X		X	X	X
seventh-ninth-free eleventh chord {1,3,4,5}														X		X	X	X	X	X
fifth-free ninth chord {1,2,3,7}															X		X	X	X	X
ninth-free eleventh chord {1,3,4,5,7}																X			X	X
compact ninth chord {1,2,3,5,7}																	X		X	X
seventh-free eleventh chord {1,2,3,4,5}																		X	X	X
compact eleventh chord {1,2,3,4,5,7}																			X	X
compact thirteenth chord {1,2,3,4,5,6,7}																				X



[R. Wille & R. Wille-Henning, « Towards a Semantology of Music », ICCS 2007, Springer, 2007]

How to reduce the combinatorial explosion?



• T. Schlemmer, S. E. Schmidt, « A formal concept analysis of harmonic forms and interval structures », *Annals of Mathematics and Artificial Intelligence* 59(2), 241– 256 (2010)

A lattice structure on intervals

Musical context $\mathbb{K} = (\mathcal{H}(\mathbb{T}_n), \mathbb{Z}_n, R)$:

- $G = \mathcal{H}(\mathbb{Z}_n)$ = objects = harmonic forms (equivalent classes up to a transposition),
- $M = \mathbb{Z}_n$ = attributes = intervals,
- R = occurrence of an interval in an harmonic form.

Example: 7-tet $\mathbb{T}_7$ (C, D, E, F, G, A, B)
Intervals = unison (0), second (1), third (2), fourth (3).

Reducing a concept lattice using congruences

Joint work with Carlos Agon and Moreno Andreatta (ICSS 2018).

Congruence: equivalence relation θ on a lattice $\mathcal{L}$, compatible with join and meet, i.e. $(\theta(a, b)$ and $\theta(c, d)) \Rightarrow (\theta(a \vee c, b \vee d)$ and $\theta(a \wedge c, b \wedge d))$, for all $a, b, c, d \in \mathcal{L}$.

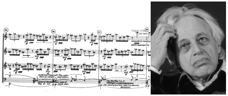
Quotient lattice: $\mathcal{L}/\theta$

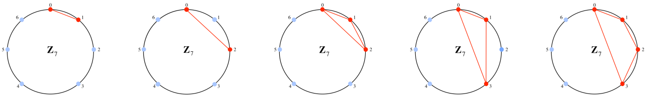
Example: congruence grouping the most common harmonic forms in a same equivalence class.

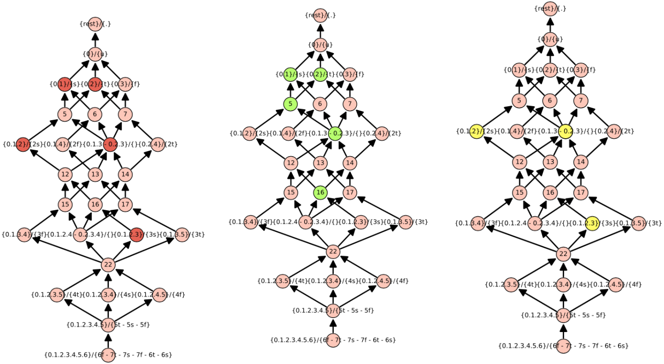
Harmonico-morphological descriptors:

- Musical piece $\mathcal{M}$, harmonic system $\mathbb{T}_{\mathcal{M}}$, concept lattice $\mathbb{C}(\mathcal{M})$
- $H_{\mathbb{C}}^{\mathcal{M}}$: formal concepts corresponding to the harmonic forms in $\mathcal{M}$
  - θ grouping all formal concepts in $H_{\mathbb{C}}^{\mathcal{M}}$ into one same class;
  - θ_δ grouping all formal concepts in $\delta(H_{\mathbb{C}}^{\mathcal{M}})$ into one same class;
  - θ_ε grouping all formal concepts in $\varepsilon(H_{\mathbb{C}}^{\mathcal{M}})$ into one same class.
- Proposed harmonic descriptors: quotient lattices $\mathbb{C}(\mathcal{M})/\theta$, $\mathbb{C}(\mathcal{M})/\theta_\delta$, and $\mathbb{C}(\mathcal{M})/\theta_\varepsilon$.

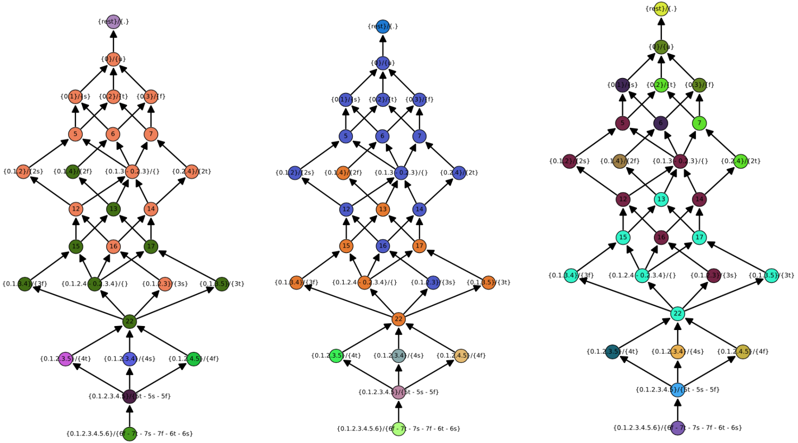
Example: Ligeti's String Quartet No. 2 (M2 Pierre Mascarade)



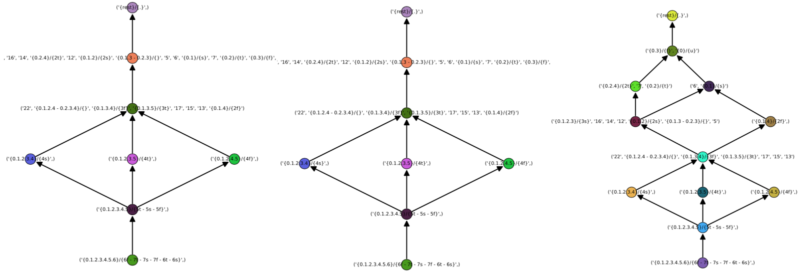




Formal concepts associated with the harmonic forms found in $H^{\mathcal{M}}$: $H^{\mathcal{M}}_{\mathbb{C}}$ (red), dilation $\delta(H^{\mathcal{M}}_{\mathbb{C}})$ (green), and erosion $\varepsilon(H^{\mathcal{M}}_{\mathbb{C}})$ (yellow).



Congruence relations θ, θ_δ, and θ_ε on $\mathbb{C}(\mathcal{M})$ (7-tet) generated by: $H_{\mathbb{C}}^{\mathcal{M}}$, $\delta(H_{\mathbb{C}}^{\mathcal{M}})$, and $\varepsilon(H_{\mathbb{C}}^{\mathcal{M}})$.



Quotient lattices: $\mathbb{C}(\mathcal{M})/\theta$, $\mathbb{C}(\mathcal{M})/\theta_\delta$, and $\mathbb{C}(\mathcal{M})/\theta_\varepsilon$.

Interpretation:

- Dilations and erosions of the set of formal concepts provide upper and lower bounds of the description.
- Congruences provide a structural summary of the harmonic forms.
- Proposed descriptors = good representative of $\mathcal{M}$, since they preserve the intervallic structures and provide compact summaries, which would allow for comparison between musical pieces.

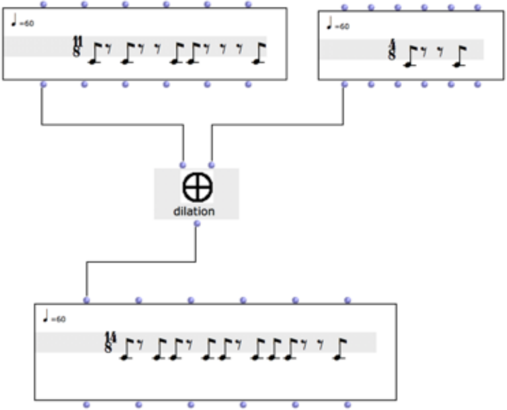
Mathematical morphology on rhythms, melodies and spatial representations

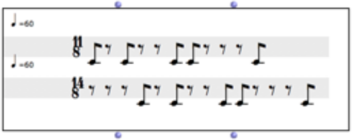
- Object = rhythm
- Structuring element = rhythm
- Dilation via time translation
- Concatenation

Future work:

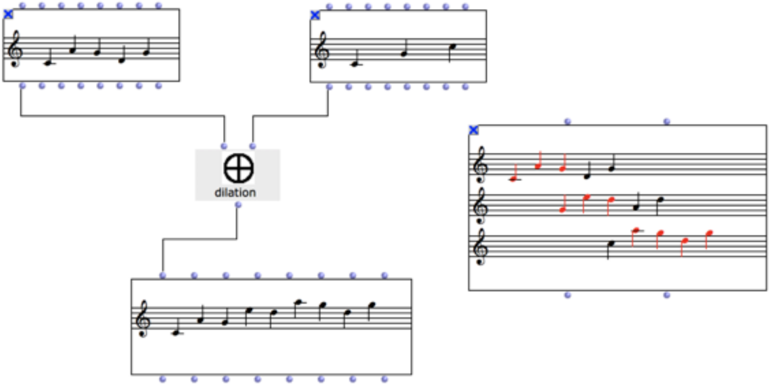
- Melodies.
- Spatial and discrete representations:
  - piano roll
  - Tonnetz
  - simplicial simplex
- Applications:
  - analysis,
  - generation,
  - comparison...

Dilation





Extension to melodies



Conclusion

- Algebraic framework of mathematical morphology.
- Strong properties.
- Natural links with logics.
- Applies in different frameworks (many types of logics, fuzzy sets, bipolarity, graphs and hypergraphs, formal concept analysis...).
- Knowledge representation.
- Reasoning (on preferences, on beliefs, on spatial information...).
- Spatial reasoning and image understanding.
- Other applications (e.g. music).