Hereditarily Homology-Simple Sets and Homology Critical Kernels of Binary Images on Sets of Convex Polytopes

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Abstract

We define a *binary image* to be a mapping $\mathbb{I} : \mathsf{X} \to \{0, 1\}$ in which X is a set of nonempty sets (e.g., a set of cubical voxels) in a Euclidean space and $\mathbb{I}^{-1}[1]$ is finite: We say $\mathbb{I}$ is a *binary image on* X and call each element of $\mathbb{I}^{-1}[1]$ a *1* of $\mathbb{I}$. For any set S of 1s of $\mathbb{I}$ we use the term S-*intersection* to mean a nonempty set that is the intersection of a nonempty subset of S. Thus an S-*intersection* is either an element of S or a nonempty intersection of two or more elements of S.

Let D be any set of 1s of a binary image $\mathbb{I}$. If the inclusion $\bigcup(\mathbb{I}^{-1}[1] \setminus \mathsf{D}) \to \bigcup \mathbb{I}^{-1}[1]$ induces homology isomorphisms in all dimensions, then we say D is *homology-simple* in $\mathbb{I}$. If every subset of D is homology-simple in $\mathbb{I}$, then we say D is *hereditarily* homology-simple in $\mathbb{I}$.

A local characterization of hereditarily homology-simple sets can be useful for designing parallel thinning algorithms or for checking the topological soundness of proposed parallel thinning algorithms. When $\mathbb{I}$ is a binary image on the grid cells of a Cartesian grid of dimension ≤ 4, it can be deduced from results of Bertrand and Couprie that the sets D of 1s that are hereditarily homology-simple in $\mathbb{I}$ can be locally characterized as follows in terms of Bertrand's concept of *critical kernel*:

- A set $\mathsf{D} \subseteq \mathbb{I}^{-1}[1]$ is hereditarily homology-simple in $\mathbb{I}$ if and only if every D-intersection in $\mathbb{I}$'s critical kernel is a subset of a 1 of $\mathbb{I}$ that is not in D.

After discussing this characterization and some of its consequences, we will explain how we can generalize it to a local characterization of hereditarily homology-simple sets of 1s in any binary image $\mathbb{I}$ on an arbitrary set of convex polytopes of any dimension. To do this, we need only replace "$\mathbb{I}$'s critical kernel" in the above characterization with "$\mathbb{I}$'s homology critical kernel". We define the latter to be the set of all $\mathbb{I}^{-1}[1]$-intersections c for which the intersection of c with the union of the 1s of $\mathbb{I}$ that do not contain c either is empty or is disconnected or has non-trivial homology in some positive dimension.