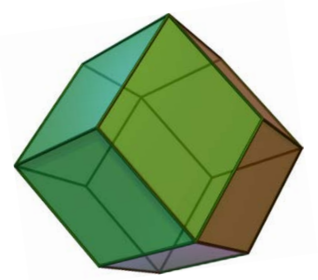
Hereditarily Homology-Simple Sets and Homology Critical Kernels of Binary Images on Sets of Convex Polytopes

T. Yung Kong
Computer Science Department
Queens College, CUNY
Flushing, NY 11367, U.S.A.

ykong@cs.qc.cuny.edu
or yung_kong@qc.edu

What is This Talk About (1)?

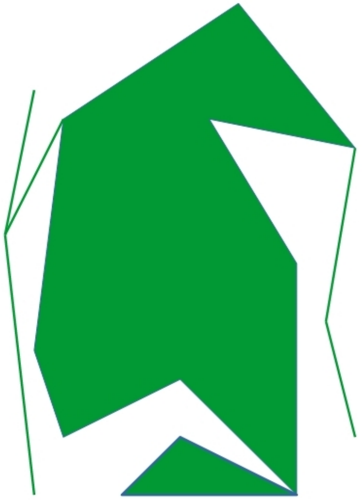
A ***convex polytope*** is a set that is the convex hull of a finite set of points in some Euclidean space $\mathbf{R}^n$.



A Convex Polytope

A ***polyhedron*** is a set that is the union of a finite collection of convex polytopes in a Euclidean space.

- The union of any finite collection of polyhedra is a polyhedron.
- The intersection of any finite collection of polyhedra is a polyhedron.



A 2D Polyhedron

- This talk will present ***homology-critical*** kernels, which are a variant of Bertrand's *critical* kernels: When dealing with sets of grid cells of a 2D, 3D, or 4D Cartesian grid, *homology-critical* and *critical* are equivalent.

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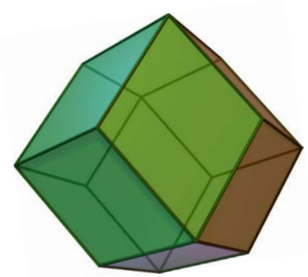

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- This talk will present ***homology-critical*** kernels, which are a variant of Bertrand's *critical* kernels: When dealing with sets of grid cells of a 2D, 3D, or 4D Cartesian grid, *homology-critical* and *critical* are equivalent.
- Many results about *critical* kernels of such sets become valid for sets of *arbitrary convex polytopes of any dimension* (and, more generally, sets of *arbitrary acyclic polyhedra whose nonempty intersections are acyclic*) if they are restated as results about *homology-critical* kernels.

- Below is an (extremely simple) 2D example of a set of polyhedra to which the main results of this talk would apply.
- While these five 2D polyhedra have disjoint interiors, our main results are also valid for collections of polyhedra whose interiors overlap.

82 *Chapter 4. Topological Digital Spaces*

From:
G. T. Herman,
Geometry of Digital Spaces,
Birkhäuser, 1998.

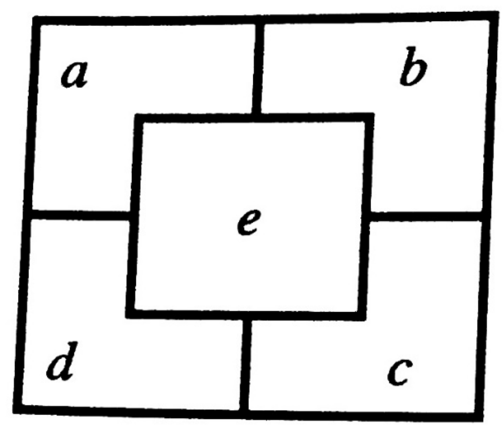
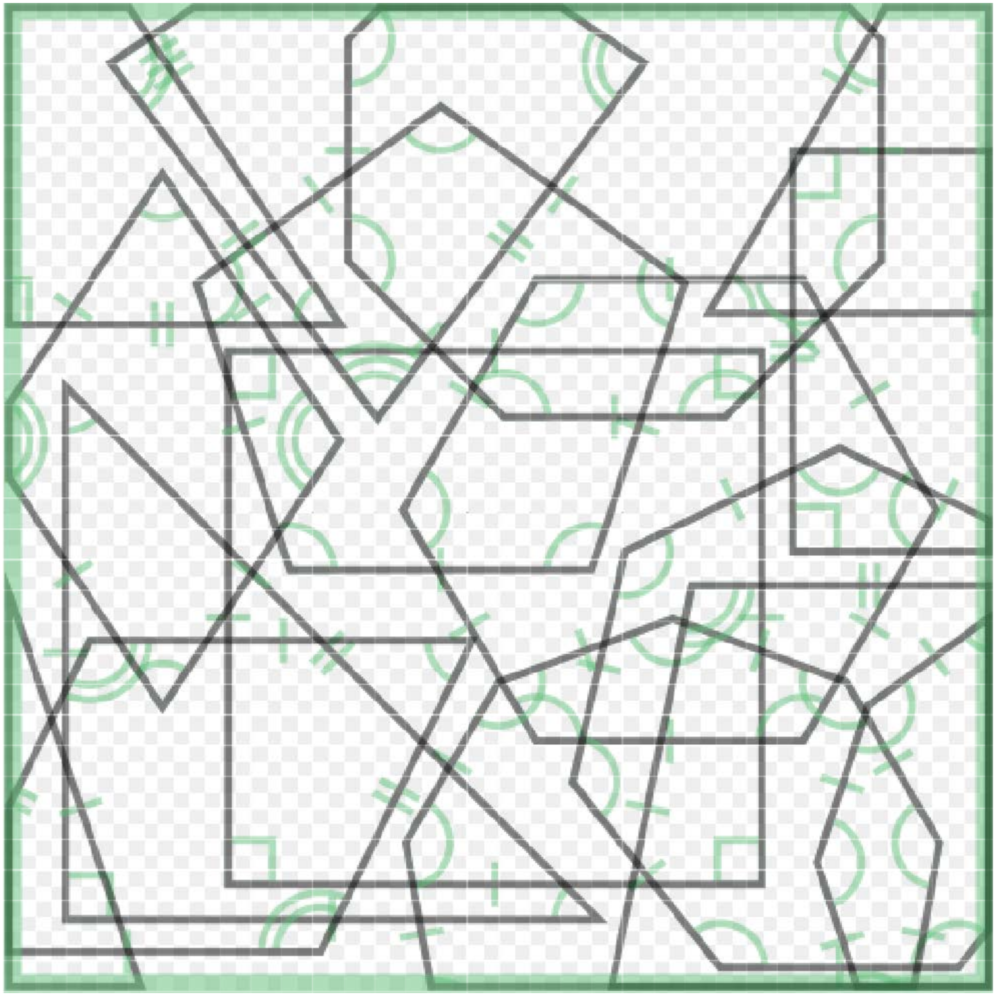


Figure 4.1.1. A simple digital space.

From Wikimedia Commons, the free media repository
File: Polygons bundle-01.svg   Date: Aug. 11, 2018   Author: Matt Grünewald
https://creativecommons.org/licenses/by-sa/4.0/deed.en

Another 2D example of a set of polyhedra to which the main results of this talk would apply:

- The polyhedra here are the 2D convex polytopes bounded by the gray lines.
- The green parts of this drawing are irrelevant.



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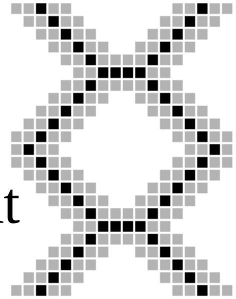

A ***thinning algorithm*** simplifies a binary image by reducing its foreground to a thin "skeleton" in a "topology-preserving" way.

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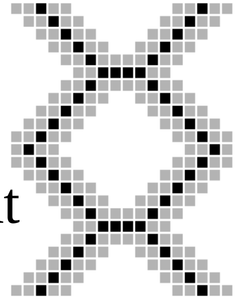

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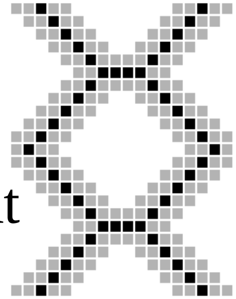

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A methodology due to Bertrand and Couprie, based on critical kernels, designs parallel thinning algorithms that always satisfy this requirement.

For binary images on *grid cells of a* 2*D*, 3*D*, *or* 4*D Cartesian grid*, a fundamental theorem of Bertrand and Couprie relating to critical kernels provides a useful local necessary and sufficient condition for every subset of a given set of image elements to be homology-simple in $\mathcal{F}$.

- Our main result (Main Theorem 2) substitutes *homology-critical* for *critical* in a statement of this theorem and so gives an analogous necessary and sufficient condition that is valid for binary images on sets of arbitrary convex polytopes of any dimension—even if some of the polytopes have overlapping interiors—and, still more generally, sets of arbitrary acyclic polyhedra whose nonempty intersections are acyclic.

Binary Images and Their Foreground Polyhedra

Let $\mathcal{G}$ be a set of polyhedra—e.g., $\mathcal{G}$ may be a set of grid cells of a nD Cartesian grid—and let $\mathbb{I} : \mathcal{G} \rightarrow \{0, 1\}$ be such that $\mathbb{I}^{-1}[\{1\}]$ is finite. Then:

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- The function $\mathbb{I}$ will be called a ***binary image*** on $\mathcal{G}$.
- If $P \in \mathcal{G}$ and $\mathbb{I}(P) = 1$, then we say P is a **1** of $\mathbb{I}$.
- If $P \in \mathcal{G}$ and $\mathbb{I}(P) = 0$, then we say P is a **0** of $\mathbb{I}$.

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- The ***foreground*** of $\mathbb{I}$ is the set $\mathbb{I}^{-1}[\{1\}]$ — i.e., the set of all **1**s of $\mathbb{I}$. This set will be denoted by $\mathcal{F}_{\mathbb{I}}$.
- The ***foreground polyhedron*** of $\mathbb{I}$ is the set $\bigcup \mathcal{F}_{\mathbb{I}} = \bigcup \mathbb{I}^{-1}[\{1\}]$.

0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	0	0
0	1	1	1	0	0	0	1	1	0	0
0	1	0	0	1	1	0	1	1	1	0
0	1	0	0	0	1	0	1	0	1	0
0	0	1	1	1	1	0	0	0	0	0
0	0	1	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

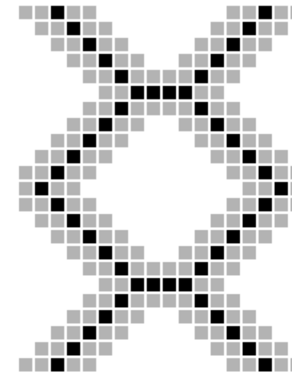
$\mathbb{I}$

$\mathcal{F}_{\mathbb{I}}$

Left:
A binary image $\mathbb{I}$ on a set of 88 grid cells of a 2D Cartesian grid.

Right:
$\mathbb{I}$'s foreground polyhedron $\mathcal{F}_{\mathbb{I}}$

Thinning Algorithms

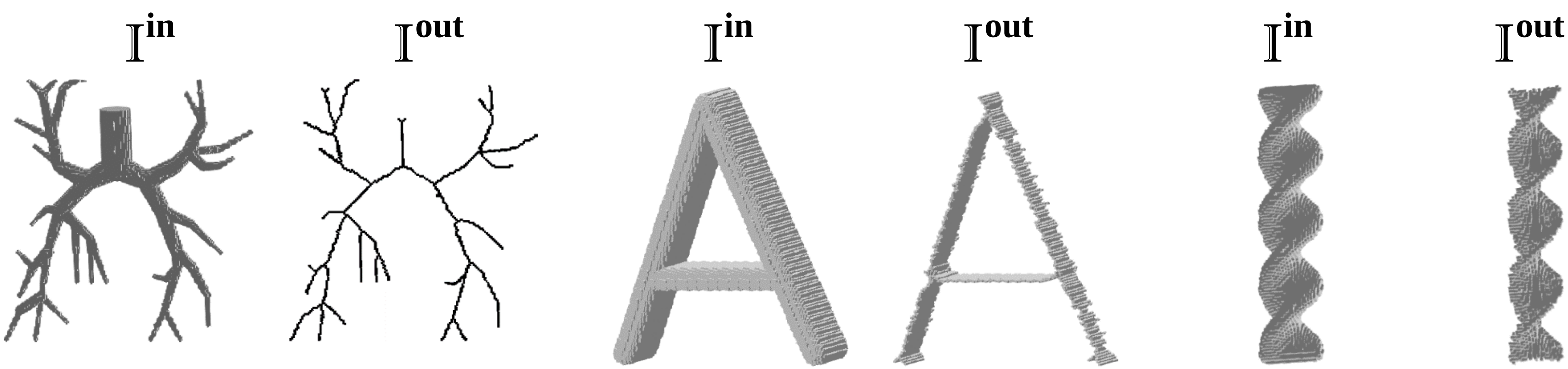


A ***thinning algorithm*** is used to transform a binary image by reducing its foreground to a thin "skeleton".

Let $\mathbb{I}^{\mathbf{in}} : \mathcal{G} \to \{0, 1\}$ and $\mathbb{I}^{\mathbf{out}} : \mathcal{G} \to \{0, 1\}$ be the *input* and *output* binary images of an *n*D thinning algorithm.

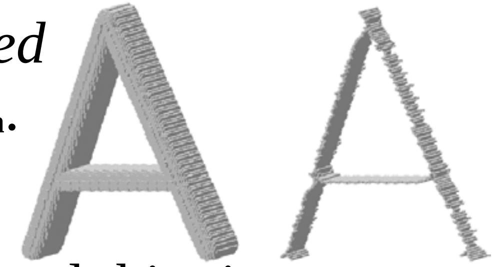
Thinning algorithms change **1**s to **0**s but never change **0**s to **1**s, so the foreground of $\mathbb{I}^{\mathbf{out}}$ is a subset of the foreground of $\mathbb{I}^{\mathbf{in}}$: $\mathcal{F}_{\mathbb{I}^{\mathbf{out}}} \subseteq \mathcal{F}_{\mathbb{I}^{\mathbf{in}}}$

3 Examples of 3D Thinning (Using Different Thinning Algorithms)
[From: C. M. Ma, S. Y. Wan, and J. D. Lee, *IEEE Transactions on Pattern Analysis and Machine Intelligence* 24 (2002) 1594 –1605]



Topological Requirements of Thinning: Homology-Simpleness

We expect 2D thinning algorithms to preserve *connected components* and *internal cavities* of $\mathcal{F}_{\mathbb{I}^{\mathbf{in}}}$.



We expect 3D thinning algorithms to preserve *connected components*, *internal cavities*, and *holes/tunnels* of $\mathcal{F}_{\mathbb{I}^{\mathbf{in}}}$.

The following condition **T** states these requirements precisely, and also generalizes them to higher-dimensional thinning:

T: The inclusion $\iota : \bigcup \mathcal{F}_{\mathbb{I}^{\mathbf{out}}} \to \bigcup \mathcal{F}_{\mathbb{I}^{\mathbf{in}}}$ must be a *homology isomorphism*—

ι must induce a group isomorphism ι_*: $H_k(\bigcup \mathcal{F}_{\mathbb{I}^{\mathbf{out}}}) \to H_k(\bigcup \mathcal{F}_{\mathbb{I}^{\mathbf{in}}})$ for all k.

Let $\mathcal{F}$ be any set of polyhedra, let $\mathcal{D} \subseteq \mathcal{F}$ and let $Q \in \mathcal{F}$. Then we say $\mathcal{D}$ is ***homology-simple*** in $\mathcal{F}$ if the inclusion $\bigcup(\mathcal{F} \setminus \mathcal{D}) \to \bigcup \mathcal{F}$ is a homology isomorphism. We say Q is ***homology-simple*** in $\mathcal{F}$ if $\{Q\}$ is.

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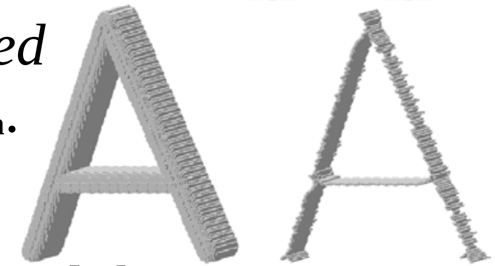

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So the topological condition **T** can also be stated as follows:

T: *The set* $\mathcal{F}_{\mathbb{I}^{\mathrm{in}}} \setminus \mathcal{F}_{\mathbb{I}^{\mathrm{out}}}$ *must be* ***homology-simple*** *in* $\mathcal{F}_{\mathbb{I}^{\mathrm{in}}}$.

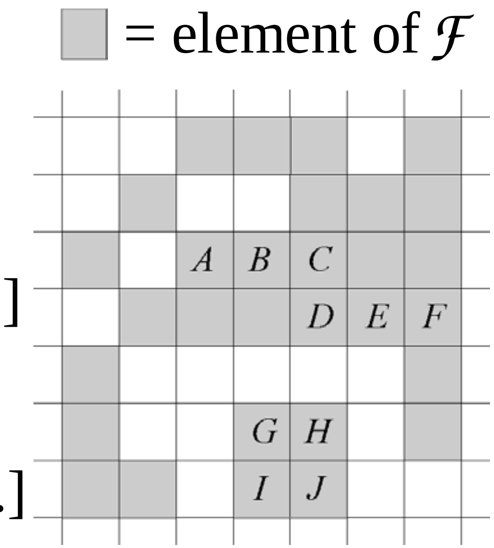
Homology-Simple Sets in the Plane

If $\mathcal{F}$ is any finite set of polyhedra ***in the plane*** $\mathbf{R}^2$ and $\mathcal{D} \subseteq \mathcal{F}$, then $\mathcal{D}$ is homology-simple in $\mathcal{F}$ if and only if ***none*** *of the following* occurs when we remove $\mathcal{D}$ from $\mathcal{F}$:

1. A component of $\bigcup\mathcal{F}$ is ***split***.
   [e.g., $\mathcal{D}$ ={*E*,*F*} is ***not*** homology-simple in $\mathcal{F}$.]

2. A component of $\bigcup\mathcal{F}$ is ***eliminated***.
   [e.g., $\mathcal{D}$ ={*G*,*H*, *I*, *J* } is ***not*** homology-simple in $\mathcal{F}$.]

3. A component of $\bigcup\mathcal{F}$ ***gains a new internal cavity***. [e.g., $\mathcal{D}$ ={*C*} is ***not*** homology-simple in $\mathcal{F}$.]



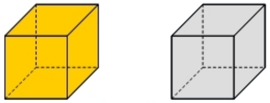
4. A component of $\bigcup\mathcal{F}$ ***loses an internal cavity*** when that internal cavity is merged with another cavity or merged with the component's outside. [e.g., {*A*} and {*B*, *C*, *D*} are ***not*** homology-simple in $\mathcal{F}$.]

Homology-Simple Sets in $\mathbf{R}^n$

More generally, if $\mathcal{F}$ is a set of polyhedra in $\mathbf{R}^n$ and $\mathcal{D} \subseteq \mathcal{F}$, then $\mathcal{D}$ is homology-simple in $\mathcal{F}$ just if there is no $k \le n$ such that removal of $\mathcal{D}$ from $\mathcal{F}$ ***splits or eliminates*** a class of homologous k-dimensional cycles.

[Two k-cycles z and z' in a set X are said to be ***homologous*** (in X) just if there exists a $(k+1)$-chain c in X such that the boundary of c is $z - z'$.]

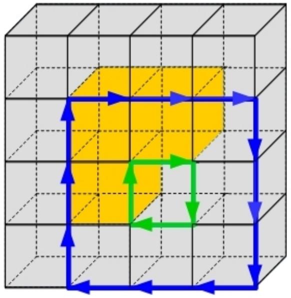
$\mathcal{D}$ is homology-simple in $\mathcal{F}$ just if ***neither*** of the following is true:



Element of $\mathcal{D}$

Element of $\mathcal{F} \setminus \mathcal{D}$

1. For some $k \le n$, $\exists$ ***non***-homologous k-cycles of $\bigcup(\mathcal{F} \setminus \mathcal{D})$ that are homologous in $\bigcup\mathcal{F}$.



Deletion of $\mathcal{D}$ ***splits*** the class of 1-cycles of $\bigcup\mathcal{F}$ that are homologous to the **blue** and the **green** 1-cycles: This "*creates a hole*".

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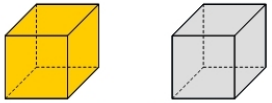

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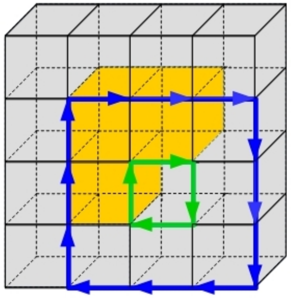
2. For some $k \leq n$, $\exists$ a k-cycle in $\bigcup\mathcal{F}$ that is ***not*** homologous to any k-cycle in $\bigcup(\mathcal{F} \setminus \mathcal{D})$.



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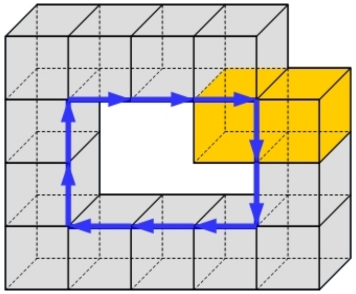
The mapping of homology classes of k-cycles induced by the inclusion $\iota : \bigcup(\mathcal{F} \setminus \mathcal{D}) \rightarrow \bigcup\mathcal{F}$ is ***not 1-1*** in case 1, ***not onto*** in case 2.



Deletion of $\mathcal{D}$ ***eliminates*** the class of 1-cycles of $\bigcup\mathcal{F}$ that are homologous to the **blue** 1-cycle: It "*eliminates a hole*".

Recall Let $\mathcal{F}$ be any finite set of polyhedra, let $\mathcal{D} \subseteq \mathcal{F}$ and let $Q \in \mathcal{F}$. Then we say $\mathcal{D}$ is ***homology-simple*** in $\mathcal{F}$ if the inclusion $\bigcup(\mathcal{F} \setminus \mathcal{D}) \to \bigcup\mathcal{F}$ is a homology isomorphism. We say Q is ***homology-simple*** in $\mathcal{F}$ if $\{Q\}$ is.

Homology-Simpleness in 2D and 3D Cartesian Grids

Most applications of binary images use binary images $\mathbb{I} : \mathcal{G} \to \{0, 1\}$ for which $\mathcal{G}$ is a set of grid cells of a 2D or 3D Cartesian grid (so that $\mathbb{I}$'s foreground $\mathcal{F}_{\mathbb{I}} = \mathbb{I}^{-1}[\{1\}]$ is a set of grid cells of the same Cartesian grid).

When $\mathcal{F}$ is a set of grid cells of a 2D or 3D Cartesian grid and $Q \in \mathcal{F}$, it can be shown that ***the following are equivalent***:

1. Q is homology-simple in $\mathcal{F}$.
2. Q is a simple element of $\mathcal{F}$ in the "traditional" (8,4) or (26,6) sense.

Regarding 2, various ***local*** characterizations of elements Q that are simple in traditional senses have been given by a number of authors —e.g., Rosenfeld (1970) in the 2D case, and Morgenthaler (1981), Tsao+Fu (1982), Saha et al. / Bertrand+Malandain (1991/2), Kong (1995), Bertrand (1996), and Bertrand+Couprie (2006) in the 3D case.

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Seq-Homology-Simpleness & Hereditary Homology-Simpleness

Let $\mathcal{F}$ be any finite set of polyhedra, and let $\mathcal{D} \subseteq \mathcal{F}$.

We say $\mathcal{D}$ is ***sequentially-homology-simple*** or ***seq-homology-simple*** in $\mathcal{F}$ if there is an enumeration $Q_1, \ldots, Q_k$ of the elements of $\mathcal{D}$ such that:

- Q_i is homology-simple in $\mathcal{F} \setminus \{Q_1, \ldots, Q_{i-1}\}$ for $1 \leq i \leq k$.

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If $|\mathcal{F}|>1$ and $\bigcup\mathcal{F}$ is acyclic but no element of $\mathcal{F}$ is homology-simple in $\mathcal{F}$ (as is possible even if $\mathcal{F}$ is a set of cubical voxels) then, for any acyclic $Q \in \mathcal{F}$, $\mathcal{F} \setminus \{Q\}$ is homology-simple but not seq-homology-simple in $\mathcal{F}$.

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$\mathcal{D}$ is seq-homology-simple in $\mathcal{F}$ $\Rightarrow$ $\mathcal{D}$ is homology-simple in $\mathcal{F}$.

as homology isomorphisms are closed under composition.

But, even in $\mathbf{R}^3$,

$\mathcal{D}$ is homology-simple in $\mathcal{F}$ $\not\Rightarrow$ $\mathcal{D}$ is seq-homology-simple in $\mathcal{F}$.

If $|\mathcal{F}|>1$ and $\bigcup\mathcal{F}$ is acyclic but no element of $\mathcal{F}$ is homology-simple in $\mathcal{F}$ (as is possible even if $\mathcal{F}$ is a set of cubical voxels) then, for any acyclic $Q\in\mathcal{F}$, $\mathcal{F}\setminus\{Q\}$ is homology-simple but not seq-homology-simple in $\mathcal{F}$.

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Seq-Homology-Simpleness & Hereditary Homology-Simpleness

Let $\mathcal{F}$ be any finite set of polyhedra, and let $\mathcal{D} \subseteq \mathcal{F}$.
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When $\mathcal{F}$ is a set of grid cells of a 2D or 3D Cartesian grid and $Q \in \mathcal{F}$, since
Q is homology-simple in $\mathcal{F}$ $\Leftrightarrow$ Q is simple in $\mathcal{F}$ in the "traditional" sense
"$\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$" can be understood ***purely in terms of simpleness in the traditional* (8,4) or (26,6) *sense***!

Digression: (4,8) or (6,26)-Simple 1s and Homology-*Co*simple Sets

Let $\mathbb{I} : \mathcal{G} \to \{0, 1\}$ be a binary image on a collection $\mathcal{G}$ of polyhedra, let $\mathcal{D} \subseteq \mathcal{F}_{\mathbb{I}} = \mathbb{I}^{-1}[\{1\}]$ and let $Q \in \mathcal{F}_{\mathbb{I}}$. Thus $\mathcal{G} \backslash \mathcal{F}_{\mathbb{I}} = \mathbb{I}^{-1}[\{0\}]$.

Then we say $\mathcal{D}$ is ***homology-cosimple*** in $\mathcal{F}_{\mathbb{I}}$ if $\mathcal{D}$ is homology-simple in $(\mathcal{G} \backslash \mathcal{F}_{\mathbb{I}}) \cup \mathcal{D}$. We say Q is ***homology-cosimple*** in $\mathcal{F}_{\mathbb{I}}$ if $\{Q\}$ is.
We say $\mathcal{D}$ is ***hereditarily*** homology-cosimple in $\mathcal{F}_{\mathbb{I}}$ if ***every*** subset of $\mathcal{D}$ is.

- When $\mathcal{G}$ is the set of all grid cells of a 2D or 3D Cartesian grid and $Q \in \mathcal{F}_{\mathbb{I}}$, it can be shown that ***the following are equivalent***:
  1. Q is homology-cosimple in $\mathcal{F}_{\mathbb{I}}$.
  2. Q is a simple element of $\mathcal{F}_{\mathbb{I}}$ in the traditional (4,8) or (6,26) sense.

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  1. Q is homology-cosimple in $\mathcal{F}_{\mathbb{I}}$.
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- When $\mathcal{G}$ is a locally finite collection of convex polytopes (or, more generally, acyclic polyhedra whose nonempty intersections are acyclic), the local characterization (in terms of homology-critical kernels) of hereditarily homology-*simple* sets $\mathcal{D}$ given by our Main Thm. 2 implies a local characterization of hereditarily homology-*cosimple* sets $\mathcal{D}$, since it can be shown that: $\mathcal{D}$ is hereditarily homology-cosimple in $\mathcal{F}_{\mathbb{I}}$

$\Leftrightarrow$ $\mathcal{D}$ is hereditarily homology-simple in $(\mathcal{G} \setminus \mathcal{F}_{\mathbb{I}}) \cup \mathcal{D}$

Recall

Thinning algorithms must satisfy the following topological requirement:

T: *The set* $\mathcal{F}_{\mathbb{I}^{\mathbf{in}}} \setminus \mathcal{F}_{\mathbb{I}^{\mathbf{out}}}$ *must be homology-simple in* $\mathcal{F}_{\mathbb{I}^{\mathbf{in}}}$.

Pseudocode of a Typical Parallel Thinning Algorithm

1. $\mathbb{I} = \mathbb{I}^{\mathbf{in}}$
2. **while** the termination condition is *not* satisfied **do**
3. $\quad\mathcal{D}$ = a subset of $\mathcal{F}_{\mathbb{I}}$ that is *hereditarily homology-simple* in $\mathcal{F}_{\mathbb{I}}$
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  - *The shape of* $\mathbb{I}^{out}$*'s foreground should reflect that of* $\mathbb{I}^{in}$*'s foreground.*
  - $\mathbb{I}^{out}$*'s foreground should be well centered relative to* $\mathbb{I}^{in}$*'s foreground.*
  - $\mathbb{I}^{out}$*'s foreground should be very thin.*

Such requirements are very important, but are not the focus of this talk.

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- For the topology preservation condition **T** to be satisfied, the set $\mathcal{D}$ that is deleted at each iteration need only be homology-simple.
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- For the topology preservation condition **T** to be satisfied, the set $\mathcal{D}$ that is deleted at each iteration need only be homology-simple.
- But algorithms in which $\mathcal{D}$ is *hereditarily* homology-simple at all iterations are more common.
- For images on Cartesian grids, methodologies that *verify* or *ensure* $\mathcal{D}$ is always hereditarily homology-simple have been developed since the '70s by, e.g., Rosenfeld, Ronse, Hall (2D), Bertrand, Ma, Bertrand & Couprie.

Critical Kernels and $\mathcal{F}$-⋂s ($\mathcal{F}$-Intersections)

Critical kernels, introduced by Bertrand (2005)—and extensively used and studied by Bertrand and Couprie—provide a powerful methodology for developing parallel thinning algorithms each of whose iterations is guaranteed to delete a hereditarily homology-simple set.

- Suppose for example that $\mathcal{K}_I$ is a subset of the foreground $\mathcal{F}_I$ at some iteration and (to satisfy non-topological requirements) we wish to *preserve* $\mathcal{K}_I$.
- We can use the critical kernel of $\mathcal{F}_I$ to find a relatively large set $\mathcal{D}$ of elements of $\mathcal{F}_I \setminus \mathcal{K}_I$ that is hereditarily homology-simple in $\mathcal{F}_I$, so that deletion of $\mathcal{D}$ preserves $\mathcal{K}_I$ and satisfies the topology-preservation condition.

Let $\mathcal{F}$ be any finite collection of nonempty sets. An ***$\mathcal{F}$-intersection*** or $\mathcal{F}$-⋂ is a ***nonempty*** set S such that $S = \bigcap C$ for some nonempty subcollection C of $\mathcal{F}$. Here C may consist of 1 member of $\mathcal{F}$: Each member of $\mathcal{F}$ is an $\mathcal{F}$-⋂.

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The critical kernel of a set $\mathcal{F}$ of grid cells of a Cartesian grid is determined by a set of $\mathcal{F}$-⋂s called ***critical*** $\mathcal{F}$-⋂s:

- The union of the critical kernel is exactly the union of the *critical* $\mathcal{F}$-⋂s.
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THEOREM (Bertrand & Couprie) *If $\mathcal{F}$ is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid, and $\mathcal{D} \subseteq \mathcal{F}$, then the following are equivalent:*

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Equivalently:

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Conversely, when $\mathcal{F}$ is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid *one can deduce Bertrand and Couprie's characterization of the minimal non-simple subsets of $\mathcal{F}$ from this theorem* (and the fact that a subset $\mathcal{T}$ of $\mathcal{F}$ is a minimal non-simple subset of $\mathcal{F}$ if and only if (i) $\mathcal{T}$ is **not** hereditarily homology-simple in $\mathcal{F}$, but (ii) every proper subset of $\mathcal{T}$ is hereditarily homology-simple in $\mathcal{F}$).

If $\mathcal{T}$ is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid, then *Bertrand & Couprie's characterization of minimal non-simple sets implies*

Recall: From Bertrand & Couprie's characterization of the minimal non-simple subsets of a finite set of grid cells of a 2D, 3D, or 4D Cartesian grid we can deduce:

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If $\mathcal{T}$ is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid, then *Bertrand & Couprie's characterization of minimal non-simple sets implies*

$$\exists \mathcal{F} \,.\, \mathcal{T} \text{ is a minimal non-simple set of } \mathcal{F} \quad \Leftrightarrow \quad \bigcap \mathcal{T} \neq \emptyset$$

and (using another theorem from Bertrand & Couprie (2009)) also implies

$$\exists \mathcal{F} \,.\, (\mathcal{T} \text{ is a minimal non-simple set of } \mathcal{F} \text{ and } \bigcup \mathcal{T} \text{ is not a component of } \bigcup \mathcal{F}) \quad \Leftrightarrow \quad \bigcap \mathcal{T} \text{ consists of more than just one point.}$$

Recall: From Bertrand & Couprie's characterization of the minimal non-simple subsets of a finite set of grid cells of a 2D, 3D, or 4D Cartesian grid we can deduce:

Theorem (Bertrand & Couprie) *If $\mathcal{F}$ is any finite set of grid cells of a 2D, 3D, or 4D Cartesian grid, and $\mathcal{D} \subseteq \mathcal{F}$, then the following are equivalent:*

1. *$\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$.*
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- These facts were originally proved over many years by Ronse (1988, 2D), Ma (1994, 3D), Kong (1995, 3D), and Gau & Kong (2003, 4D).

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Bertrand & Couprie's $\mathcal{F}$-critical $\mathcal{F}$-$\bigcap$s are defined in terms of ***collapsing*** of subcomplexes of the cubical complex whose set of facets is $\mathcal{F}$. *But we are going to use a more general concept*: $\mathcal{F}$-***homology-critical*** $\mathcal{F}$-$\bigcap$s.

Recall: **Theorem** (Bertrand & Couprie) *If $\mathcal{F}$ is any finite set of grid cells of a* 2*D*, 3*D, or* 4*D Cartesian grid, and $\mathcal{D} \subseteq \mathcal{F}$, then the following are equivalent:*

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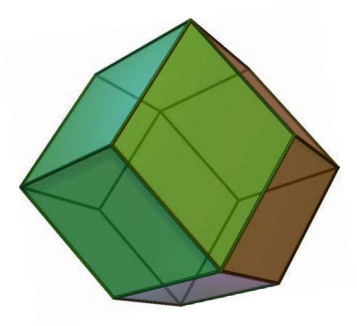
- Our main result is that ***this*** version of the theorem is valid much more generally: It is valid when $\mathcal{F}$ is *any finite set of acyclic polyhedra whose nonempty intersections are acyclic*.

  For example, it is valid when $\mathcal{F}$ is *any finite set of convex polytopes*.

Acyclic Polyhedra



A ***convex polytope*** is a set that is the convex hull of a finite set of points in some Euclidean space $\mathbf{R}^n$.

A ***polyhedron*** is a set that is the union of a finite collection of convex polytopes in a Euclidean space.

- The union of any finite collection of polyhedra is a polyhedron.
- The intersection of any finite collection of polyhedra is a polyhedron.

A set P is said to be ***acyclic*** if

1. P is nonempty and connected, and
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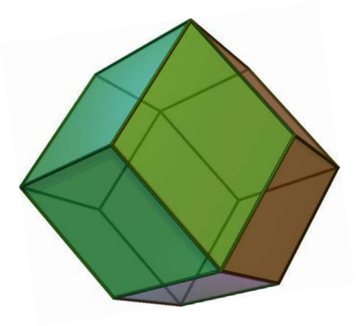

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1. P is nonempty and connected.
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3. The Euler characteristic of P is 1.

When 1 and 2 hold, 3 holds if and only if P "has no holes or tunnels" (and if and only if P is simply connected).

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In $\mathbf{R}^n$ (for any dimension n), we have that:

- Any convex polytope is acyclic; more generally, if C is any nonempty collection of convex sets such that $\bigcap C \neq \emptyset$, then $\bigcup C$ is acyclic.
- If P and Q are two acyclic polyhedra such that $P \cap Q$ is acyclic, then $P \cup Q$ is also acyclic.

We say a set $\mathcal{G}$ of polyhedra is ***good*** if $\mathcal{G}$ is finite, each member of $\mathcal{G}$ is acyclic, and every nonempty intersection of ≥ 2 members of $\mathcal{G}$ is acyclic.
Example: *Any finite set of convex polytopes is a good set of polyhedra*.

$\mathcal{F}$-Cores of $\mathcal{F}$-⋂s; $\mathcal{F}$-Homology-Critical $\mathcal{F}$-⋂s

Let $\mathcal{F}$ be any finite collection of nonempty sets. Recall that an $\mathcal{F}$-⋂ is a ***nonempty*** set S such that $S = \bigcap \mathcal{C}$ for some nonempty subcollection $\mathcal{C}$ of $\mathcal{F}$. [$\mathcal{C}$ may consist of just one member of $\mathcal{F}$: Any member of $\mathcal{F}$ is an $\mathcal{F}$-⋂!]

We now define the $\mathcal{F}$-***core*** of an $\mathcal{F}$-⋂.

This concept is very similar to Bertrand's concept of the ***core*** of a cell of a complex (but does not refer to any complex)**.**

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If C is an $\mathcal{F}$-⋂, then we define: $\mathbf{Core}_{\mathcal{F}}(C) \overset{\text{def}}{=} C \cap \bigcup\{F \in \mathcal{F} \mid F \not\supseteq C\}$

Thus: $\mathbf{Core}_{\mathcal{F}}(C)$ = the intersection of C with the union of those members of $\mathcal{F}$ that do not contain C.

We call $\mathbf{Core}_{\mathcal{F}}(C)$ the ***$\mathcal{F}$-core*** of C.

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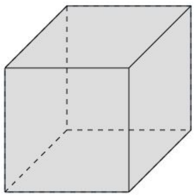
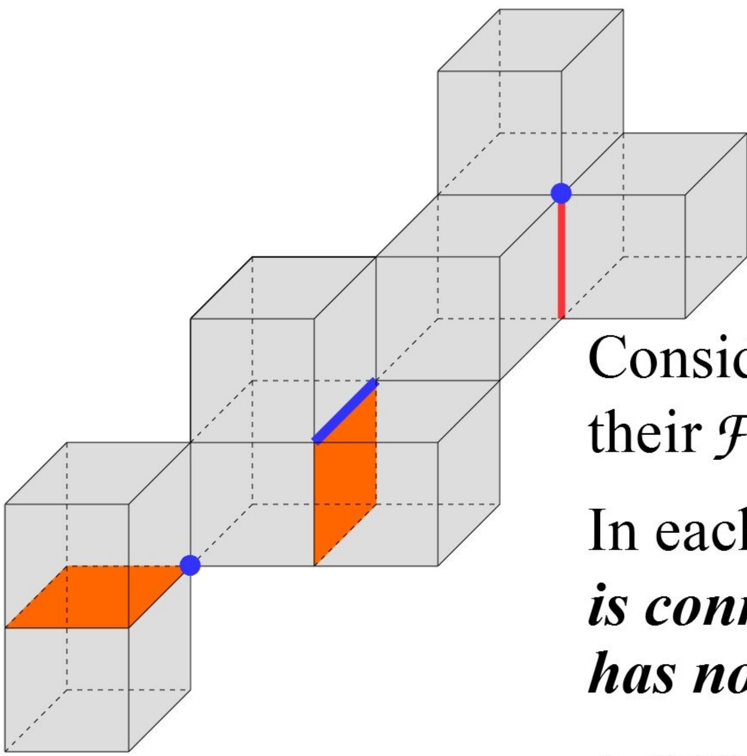
An $\mathcal{F}$-⋂ C is said to be $\mathcal{F}$-***homology-critical*** if $\mathbf{Core}_{\mathcal{F}}(C)$ is ***not*** acyclic.
Hence: An $\mathcal{F}$-⋂ is $\mathcal{F}$-homology-critical if and only if its $\mathcal{F}$-core is ∅, or is disconnected, or has nontrivial homology in some positive dimension.

Three Examples of $\mathcal{F}$-⋂s That are *NOT* $\mathcal{F}$-Homology-Critical

 = element of $\mathcal{F}$

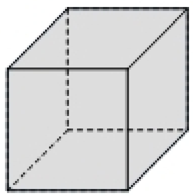


Consider the 3 **orange** $\mathcal{F}$-⋂s and their $\mathcal{F}$-cores (**colored blue**).

In each case, the $\mathcal{F}$-core ***is nonempty***, ***is connected***, ***has no hole***, and ***has no internal cavity***.

∴ ***<u>none</u>*** of these 3 $\mathcal{F}$-⋂s is $\mathcal{F}$-homology-critical!

Six Examples of $\mathcal{F}$-Homology-Critical $\mathcal{F}$-⋂s



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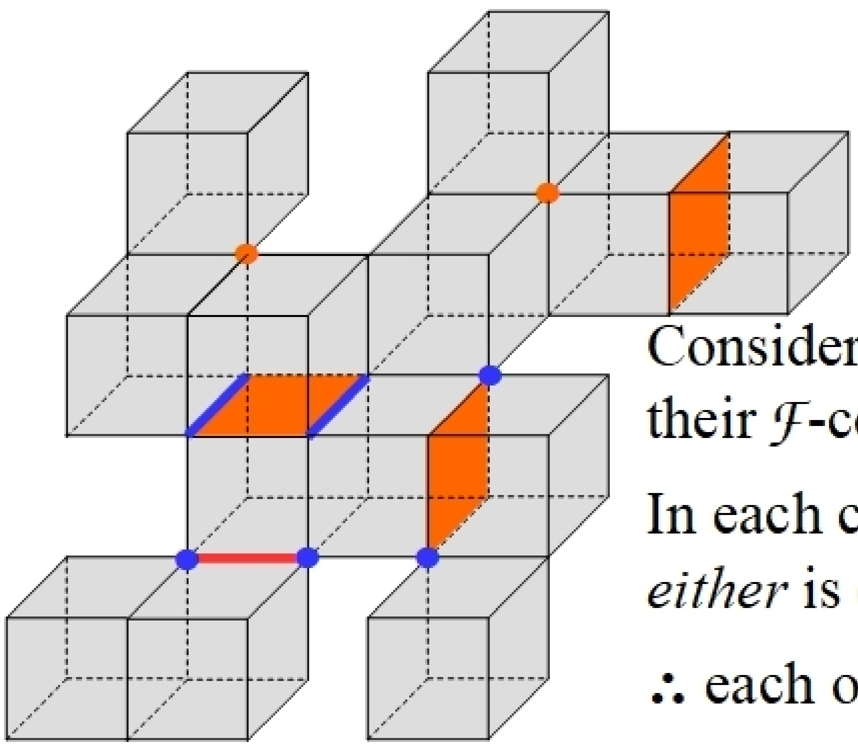
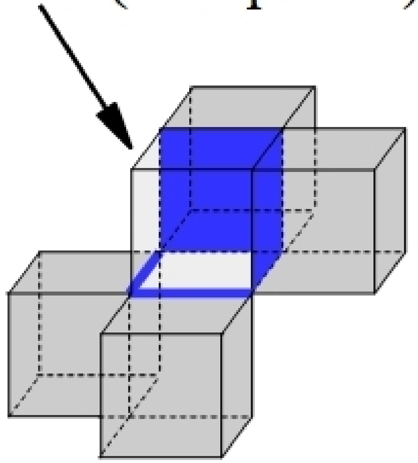


Consider the 6 **orange** $\mathcal{F}$-⋂s and their $\mathcal{F}$-cores (**colored blue**):

In each case, the $\mathcal{F}$-core *either* is empty *or* is disconnected.

∴ each of these 6 $\mathcal{F}$-⋂s is $\mathcal{F}$-***homology-critical***!

If $\mathcal{F}$ is the set of five cubes that are are shown here,
and C is ***this*** (transparent) cube

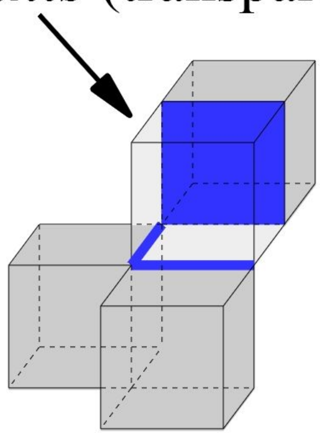


… then $\mathbf{Core}_{\mathcal{F}}(C)$ is the **blue** set.

$\mathbf{Core}_{\mathcal{F}}(C)$ *has a hole.*

∴ C ***is*** $\mathcal{F}$-homology-critical!

If $\mathcal{F}$ is the set of four cubes that are are shown here, and
C is ***this*** (transparent) cube



… then $\mathbf{Core}_{\mathcal{F}}(C)$ is the **blue** set.

$\mathbf{Core}_{\mathcal{F}}(C)$ is *nonempty*, is *connected*,
has no hole, and *has no internal cavity*.

∴ C is ***not*** $\mathcal{F}$-homology-critical!

Recall: If C is any $\mathcal{F}$-⋂, then

$$\mathbf{Core}_{\mathcal{F}}(C) \stackrel{\text{def}}{=} C \cap \bigcup\{F \in \mathcal{F} \mid F \not\supseteq C\}$$

= the intersection of C with the union of those members of $\mathcal{F}$ that do ***not*** contain C.

An $\mathcal{F}$-⋂ C is said to be ***$\mathcal{F}$-homology-critical*** if $\mathbf{Core}_{\mathcal{F}}(C)$ is ***not*** acyclic.

∴ An $\mathcal{F}$-⋂ is $\mathcal{F}$-homology-critical ***if and only if*** its $\mathcal{F}$-core is ∅, or is disconnected, or has nontrivial homology in some positive dimension.

We define the ***homology-critical kernel*** of $\mathcal{F}$ to be the set of all $\mathcal{F}$-homology-critical $\mathcal{F}$-⋂s.

Notes: 1. If $\mathcal{F}$ is a set of grid cells of a 2D, 3D, or 4D Cartesian grid, then an $\mathcal{F}$-⋂ is $\mathcal{F}$-homology-critical ***if and only if*** it is $\mathcal{F}$-critical in the sense of Bertrand and Couprie. (This follows from results established by Couprie & Bertrand (2009) and Kong (1997).)

2. If C is any $\mathcal{F}$-⋂, then it is readily confirmed that:

$$\begin{aligned}\mathbf{Core}_{\mathcal{F}}(C) &= \bigcup\{C \cap F \mid F \in \mathcal{F} \text{ and } F \not\supseteq C\}\\ &= \bigcup\{Y \mid Y \text{ is an } \mathcal{F}\text{-}\bigcap \text{ and } Y \subsetneq C\}\\ &= \text{the union of the } \mathcal{F}\text{-}\bigcap\text{s strictly contained in } C.\end{aligned}$$

$\mathbb{P}$-Homology-Simple Elements

Let $\mathcal{F}$ be a finite set of polyhedra. If $Q \in \mathcal{D} \subseteq \mathcal{F}$, then we say Q is ***$\mathbb{P}$-homology-simple*** for $\mathcal{D}$ in $\mathcal{F}$ if the following is true:

- Q is homology-simple in $\mathcal{F} \setminus \mathcal{S}$ for all $\mathcal{S} \subseteq \mathcal{D} \setminus \{Q\}$.

This definition is a straightforward generalization of a concept that was originally defined by Bertrand (1995).

Note: If $Q \in \mathcal{D}' \subseteq \mathcal{D} \subseteq \mathcal{F}$ and Q is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$, then it is evident that Q is also $\mathbb{P}$-homology-simple for $\mathcal{D}'$ in $\mathcal{F}$.

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We will see that: $\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$ ***if and only if*** every element of $\mathcal{D}$ is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$.

Now consider the subset of $\mathcal{D}$ defined by:

$$\mathbb{P}(\mathcal{D}, \mathcal{F}) = \{Q \in \mathcal{D} \mid Q \text{ is } \mathbb{P}\text{-homology-simple for } \mathcal{D} \text{ in } \mathcal{F}\}$$

From the case $\mathcal{D}' = \mathbb{P}(\mathcal{D}, \mathcal{F})$ of the above Note, we see that *every element of* $\mathbb{P}(\mathcal{D}, \mathcal{F})$ *is* $\mathbb{P}$-*homology-simple for* $\mathbb{P}(\mathcal{D}, \mathcal{F})$ *in* $\mathcal{F}$. Hence:

For any $\mathcal{D} \subseteq \mathcal{F}$, the set $\mathbb{P}(\mathcal{D}, \mathcal{F})$ is hereditarily homology-simple in $\mathcal{F}$.

A *Local* Characterization of $\mathbb{P}$-Homology-Simpleness

Our 1st main result characterizes $\mathbb{P}$-homology-simpleness ***locally***, in terms of $\mathcal{F}$-homology-critical $\mathcal{F}$-$\cap$s.

When $\mathcal{F}$ is a set of grid cells of a 2D, 3D, or 4D Cartesian grid, one can deduce this theorem from a theorem of Bertrand & Couprie (2009), since one can show [using results of Couprie & Bertrand (2009)] that in this case "$\mathcal{F}$-homology-critical" and "$\mathbb{P}$-homology-simple" are equivalent to the concepts of "critical" and "P-simple" used by Bertrand & Couprie:

A *Local* Characterization of ℙ-Homology-Simpleness

Our 1st main result characterizes ℙ-homology-simpleness ***locally***, in terms of $\mathcal{F}$-homology-critical $\mathcal{F}$-⋂s.

When $\mathcal{F}$ is a set of grid cells of a 2D, 3D, or 4D Cartesian grid, one can deduce this theorem from a theorem of Bertrand & Couprie (2009), since one can show [using results of Couprie & Bertrand (2009)] that in this case "$\mathcal{F}$-homology-critical" and "ℙ-homology-simple" are equivalent to the concepts of "critical" and "P-simple" used by Bertrand & Couprie:

MAIN THEOREM 1 *Let $\mathcal{F}$ be any finite set of polyhedra such that every $\mathcal{F}$-⋂ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:*

1. *Q is ℙ-homology-simple for $\mathcal{D}$ in $\mathcal{F}$.*
2. *Every $\mathcal{F}$-homology-critical $\mathcal{D}$-⋂ contained in Q is also contained in a member of $\mathcal{F} \setminus \mathcal{D}$.*

Note: Condition 2 ⇔ *Every $\mathcal{F}$-homology-critical $\mathcal{F}$-⋂ contained in Q is also contained in a member of $\mathcal{F} \setminus \mathcal{D}$.*

since any $\mathcal{F}$-⋂ that's not a $\mathcal{D}$-⋂ is evidently contained in a member of $\mathcal{F} \setminus \mathcal{D}$!

Attachment Set of a Polyhedron in a Set of Polyhedra

If P is a polyhedron and $\mathcal{L}$ a set of polyhedra then we define

$$\textbf{Attach}(P, \mathcal{L}) \stackrel{\text{def}}{=} P \cap \bigcup(\mathcal{L} \setminus \{P\})$$

and we call this set the $\mathcal{L}$-***attachment set*** of P. Note that:

1. **Attach**$(P, \mathcal{L})$ = **Attach**$(P, \mathcal{L} \cup \{P\})$ = **Attach**$(P, \mathcal{L} \setminus \{P\})$
2. If $P \notin \mathcal{L}$, then **Attach**$(P, \mathcal{L}) = P \cap \bigcup\mathcal{L}$.
3. If P is an inclusion-maximal member of $\mathcal{L}$, **Attach**$(P, \mathcal{L})$ = **Core**$_{\mathcal{L}}(P)$.
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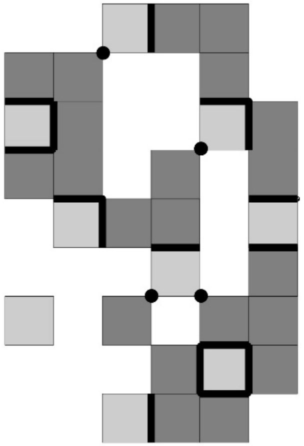
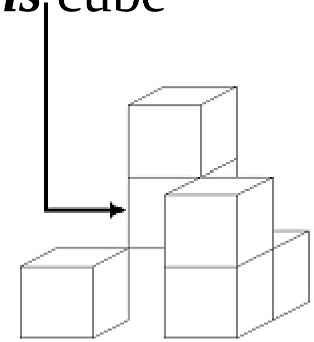
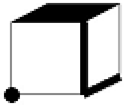
If $\mathcal{L}$ is the set of pale gray and dark gray squares on the right, then the $\mathcal{L}$-attachment set or $\mathcal{L}$-core of any *pale gray* square is the union of its black 0- and 1-faces.



If $\mathcal{L}$ is the set of 6 cubes shown below, and P is ***this*** cube



. . . then $\textbf{Attach}(P, \mathcal{L}) = \textbf{Core}_{\mathcal{L}}(P)$ is the union of the 0-, 1-, and 2-faces that are colored black here: 

In the sequel, $\mathcal{F}$ denotes a finite set of acyclic polyhedra.
(Many later results will further assume that every $\mathcal{F}$-$\bigcap$ is acyclic.)

Proposition 1: *Let $Q \in \mathcal{L} \subseteq \mathcal{F}$. Then the following are equivalent:*

(a) *Q is homology-simple in $\mathcal{L}$.* (b) **Attach**$(Q, \mathcal{L})$ *is acyclic.*

Note: If Q is ***inclusion-maximal*** in $\mathcal{L}$, then **Attach**$(Q, \mathcal{L})$ = **Core**$_{\mathcal{L}}(Q)$ and so Q is homology-simple in $\mathcal{L}$ $\Leftrightarrow$ Q is ***not*** $\mathcal{L}$-homology-critical.

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Corollary 2: *Let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then Q is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$* ***if and only if*** **Attach**$(Q, \mathcal{F} \setminus \mathcal{S})$ *is acyclic for all $\mathcal{S} \subseteq \mathcal{D}$.*

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Prop. 1 follows from the fact that **Attach**$(Q, \mathcal{L}) = Q \cap \bigcup(\mathcal{L} \setminus \{Q\})$ and results of topology: Reduced homology sequences and the Excision Thm.

$$\begin{array}{ccccccccc}
 & 0 & & \overset{\text{B}}{0?} & \cong & \overset{\text{A}}{0?} & & 0 & \\
\longrightarrow & \widetilde{H}_p(Q) & \longrightarrow & \widetilde{H}_p(Q, \textbf{Attach}(Q,\mathcal{L})) & \longrightarrow & \widetilde{H}_{p-1}(\textbf{Attach}(Q,\mathcal{L})) & \longrightarrow & \widetilde{H}_{p-1}(Q) & \longrightarrow \\
 & & & \wr\| \text{ by Excision} & & & & & \\
\longrightarrow & \widetilde{H}_p(\bigcup\mathcal{L}) & \longrightarrow & \widetilde{H}_p(\bigcup\mathcal{L}, \bigcup(\mathcal{L} \setminus\{Q\})) & \longrightarrow & \widetilde{H}_{p-1}(\bigcup(\mathcal{L} \setminus\{Q\})) & \longrightarrow & \widetilde{H}_{p-1}(\bigcup\mathcal{L}) & \longrightarrow \\
\underset{\text{D}}{\cong?} & & \underset{\text{C}}{0?} & \underset{\text{B}}{0?} & \underset{\text{C}}{0?} & & \underset{\text{D}}{\cong?} & & \underset{\text{C}}{0?}
\end{array}$$

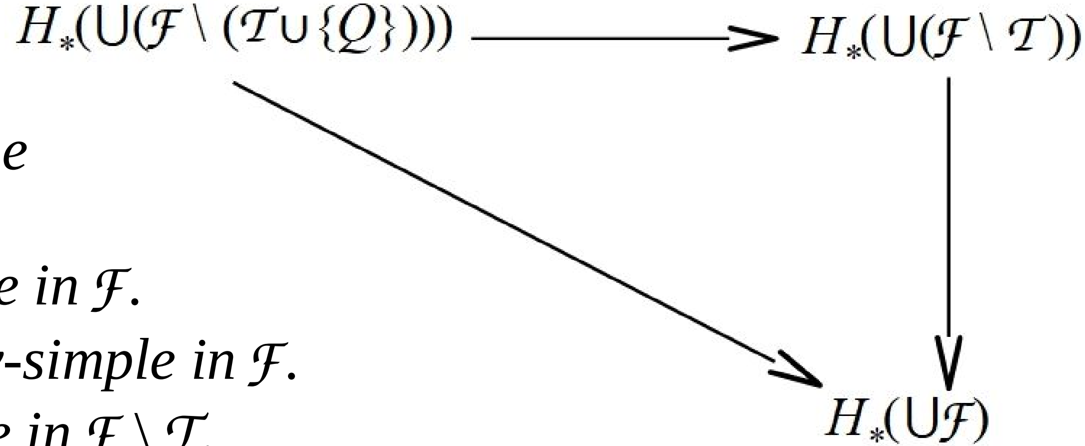
A $\Leftrightarrow$ B $\Leftrightarrow$ C $\Leftrightarrow$ D

Lemma 3: *Let $\mathcal{T} \subseteq \mathcal{F}$ and let $Q \in \mathcal{F} \setminus \mathcal{T}$. Then all of the following are true if any two are true*:

A. $\mathcal{T}$ *is homology-simple in* $\mathcal{F}$.

B. $\mathcal{T} \cup \{Q\}$ *is homology-simple in* $\mathcal{F}$.

C. Q *is homology-simple in* $\mathcal{F} \setminus \mathcal{T}$.



- The conclusion of Lemma 3 remains true—with the same proof—if we replace $\{Q\}$ and Q in **B** and **C** with *any subset* $\mathcal{T}'$ *of* $\mathcal{F} \setminus \mathcal{T}$! (But we only need the special case that is stated in the lemma.)

- From this lemma we can deduce the following previously stated fact:

$\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$

$\Leftrightarrow$ $\mathcal{D}$ is hereditarily seq-homology-simple in $\mathcal{F}$

$\Leftrightarrow$ for ***every*** enumeration $Q_1, \ldots, Q_k$ of the elements of $\mathcal{D}$ Q_i is homology-simple in $\mathcal{F} \setminus \{Q_1, \ldots, Q_{i-1}\}$ for $1 \leq i \leq k$

RECALL: **Lemma 3**: *Let* $\mathcal{S} \subseteq \mathcal{F}$ *and let* $Q \in \mathcal{F} \setminus \mathcal{S}$*. Then all of the following are true if any two are true*:

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C. $\{Q\}$ *is homology-simple in* $\mathcal{F} \setminus \mathcal{S}$.

Proposition 4: *Let* $\mathcal{D} \subseteq \mathcal{F}$*. Then the following are equivalent:*

1. $\mathcal{D}$ *is hereditarily homology-simple in* $\mathcal{F}$.
2. *For all* $Q \in \mathcal{D}$ *and* $\mathcal{S} \subseteq \mathcal{D} \setminus \{Q\}$,
   $\mathcal{S} \cup \{Q\}$ is *homology-simple in* $\mathcal{F}$ ***if*** $\mathcal{S}$ is *homology-simple in* $\mathcal{F}$.
3. *For all* $Q \in \mathcal{D}$ *and* $\mathcal{S} \subseteq \mathcal{D} \setminus \{Q\}$, $\{Q\}$ *is homology-simple in* $\mathcal{F} \setminus \mathcal{S}$.
4. *Every* $Q \in \mathcal{D}$ *is* $\mathbb{P}$*-homology-simple for* $\mathcal{D}$ *in* $\mathcal{F}$.

Proof: $2 \Rightarrow 1$ by induction, because $\emptyset$ is homology-simple in $\mathcal{F}$.

$3 \Rightarrow 2$ and $1 \Rightarrow 3$ both follow from Lemma 3.

$3 \Leftrightarrow 4$ follows from the definition of $\mathbb{P}$-*homology-simple*. //

RECALL: MAIN THEOREM 1 *Let $\mathcal{F}$ be a finite set of polyhedra such that every $\mathcal{F}$-$\bigcap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:*

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Proposition 4: *Let $\mathcal{D} \subsetneq \mathcal{F}$. Then the following are equivalent:*

1. *$\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$.* 4. *Each $Q \in \mathcal{D}$ is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$.*

A *Local* Characterization of Hereditarily Homology-Simple Sets

From Main Theorem 1 and the equivalence of statements 1 and 4 of Proposition 4 we deduce:

MAIN THEOREM 2: *Let $\mathcal{F}$ be any finite set of acyclic polyhedra such that every $\mathcal{F}$-$\bigcap$ is acyclic, and let $\mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:*

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since an $\mathcal{F}$-$\bigcap$ that's not a $\mathcal{D}$-$\bigcap$ is evidently contained in a member of $\mathcal{F} \setminus \mathcal{D}$!

If no member of $\mathcal{F}$ contains another member of $\mathcal{F}$, then condition 2 implies that

no member of $\mathcal{D}$ is $\mathcal{F}$-homology-critical

or, equivalently, that

every member of $\mathcal{D}$ is homology-simple in $\mathcal{F}$

Recall: **MAIN THM. 1** *Let $\mathcal{F}$ be any finite set of polyhedra such that every $\mathcal{F}$-$\bigcap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:*

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Notation: If C is any $\mathcal{F}$-$\bigcap$, we define $\mathcal{F}_C \overset{\text{def}}{=} \{F \in \mathcal{F} \mid F \supseteq C\}$ (so $\bigcap \mathcal{F}_C = C$).

We now restate Main Theorems 1 & 2 in terms of these sets $\mathcal{F}_C$:

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(ii) *$\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$* ***if and only if*** *there is no $\mathcal{F}$-homology-critical $\mathcal{F}$-$\bigcap$ C such that $\mathcal{F}_C \subseteq \mathcal{D}$.*

Another Version of the Main Results When $\mathcal{F}$ is Strongly Normal (SN)

Let $\mathcal{F}$ be any finite collection of polyhedra such that every $\mathcal{F}$-$\bigcap$ is acyclic.
Then for every $P \in \mathcal{F}$, we define: $\mathcal{N}^*(P,\mathcal{F}) = \{ F \in \mathcal{F} \setminus \{P\} \mid F \cap P \neq \emptyset \}$
Each member of $\mathcal{N}^*(P,\mathcal{F})$ will be called an ***$\mathcal{F}$-neighbor*** of P.

We say $\mathcal{F}$ is ***strongly normal* (SN)** if the following is true:

- $\forall P \in \mathcal{F}$. P intersects every nonempty intersection of two or more $\mathcal{F}$-neighbors of P.

Equivalently: • $\forall P \in \mathcal{F}$. P intersects every $\mathcal{N}^*(P,\mathcal{F})$-$\bigcap$.

Motivating Example: Any set of grid cells of a Cartesian grid (of any dimension) is SN.

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Motivating Example: Any set of grid cells of a Cartesian grid (of any dimension) is SN.

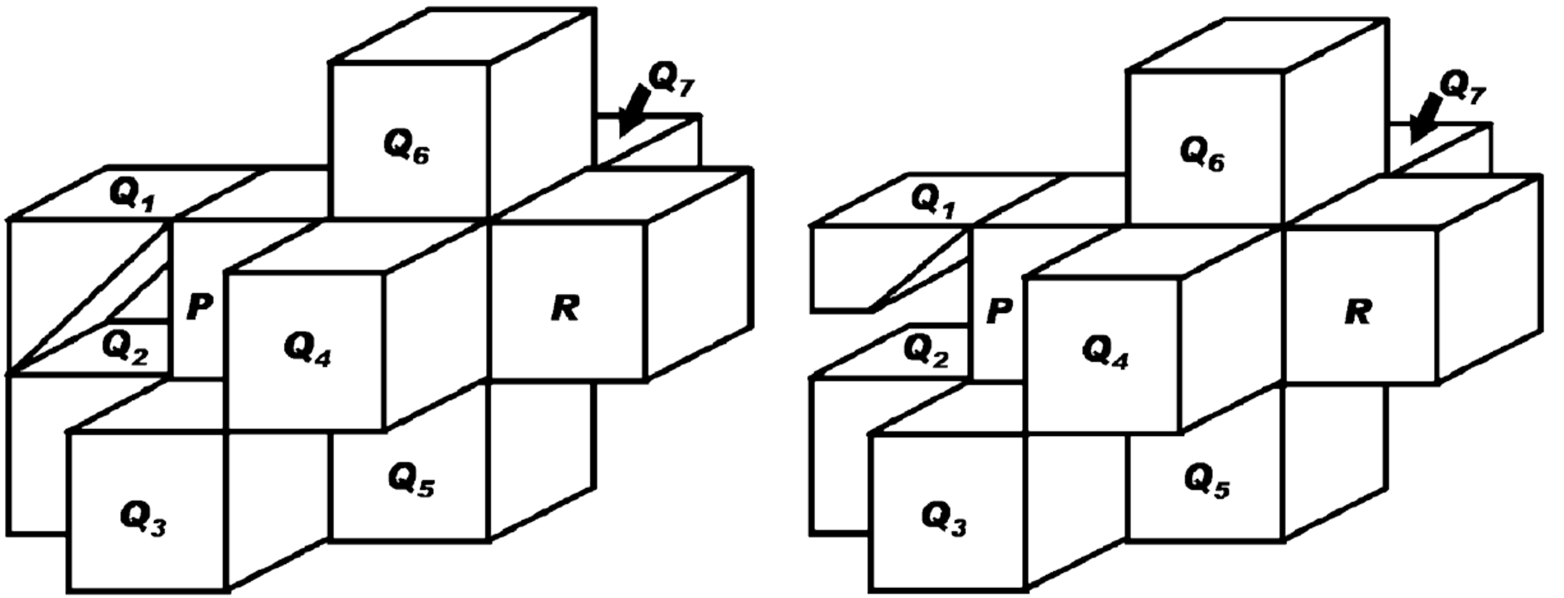
- $\mathcal{F}$ is SN $\Rightarrow$ every subcollection of $\mathcal{F}$ is strongly normal
- $\mathcal{F}$ is SN $\Leftrightarrow$ $\mathcal{F}$ is a Helly family of order 2
- $\mathcal{F}$ is SN $\Leftrightarrow$ the collection of all $\mathcal{F}$-⋂s is strongly normal

SN collections were studied in several papers (1998 – 2007) by Saha, Rosenfeld, and others (Majumder, Brass, Kong).

<u>If $\mathcal{F}$ is SN, we can state Main Theorems 1 and 2 in terms of "cliques" in $\mathcal{F}$</u>.

A ***Non***-Strongly Normal Collection, and a Strongly Normal Collection

From: T.Y. Kong, P.K. Saha, A. Rosenfeld, Strongly normal sets of contractible tiles in *N* dimensions, *Pattern Recognition* **40** (2007) 530 – 543.



(a) (b)

In (a), $\mathcal{F}$ ={**P**, **Q**$_1$, ..., **Q**$_7$, **R**} is ***not*** a strongly normal collection.

In (b), {**P**, **Q**$_1$, ..., **Q**$_7$, **R**} is a strongly normal collection.

Reason:

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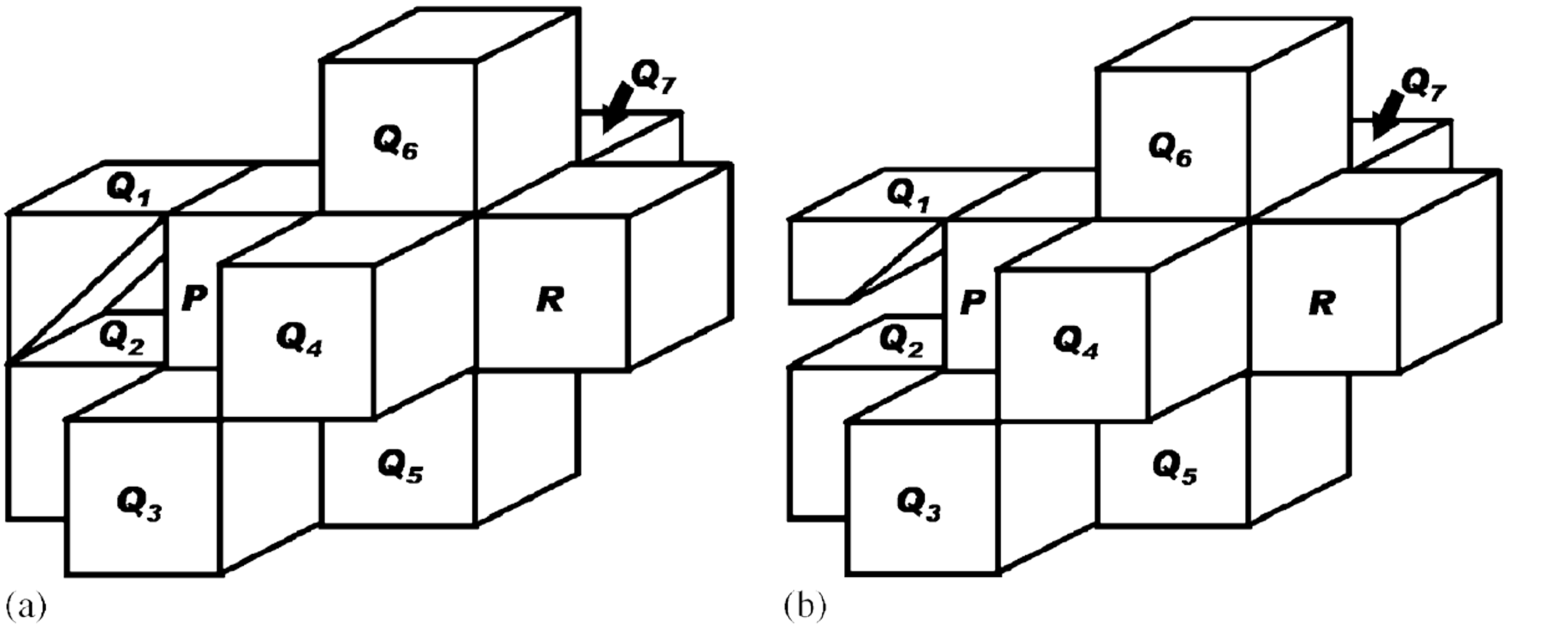

In (a), $\mathcal{F} = \{\mathbf{P}, \mathbf{Q}_1, \ldots, \mathbf{Q}_7, \mathbf{R}\}$ is ***not*** a strongly normal collection.

Reason: $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are $\mathcal{F}$-neighbors of $\mathbf{P}$, $\mathbf{Q}_1 \cap \mathbf{Q}_2 \neq \emptyset$, but $\mathbf{P} \cap \mathbf{Q}_1 \cap \mathbf{Q}_2 = \emptyset$.

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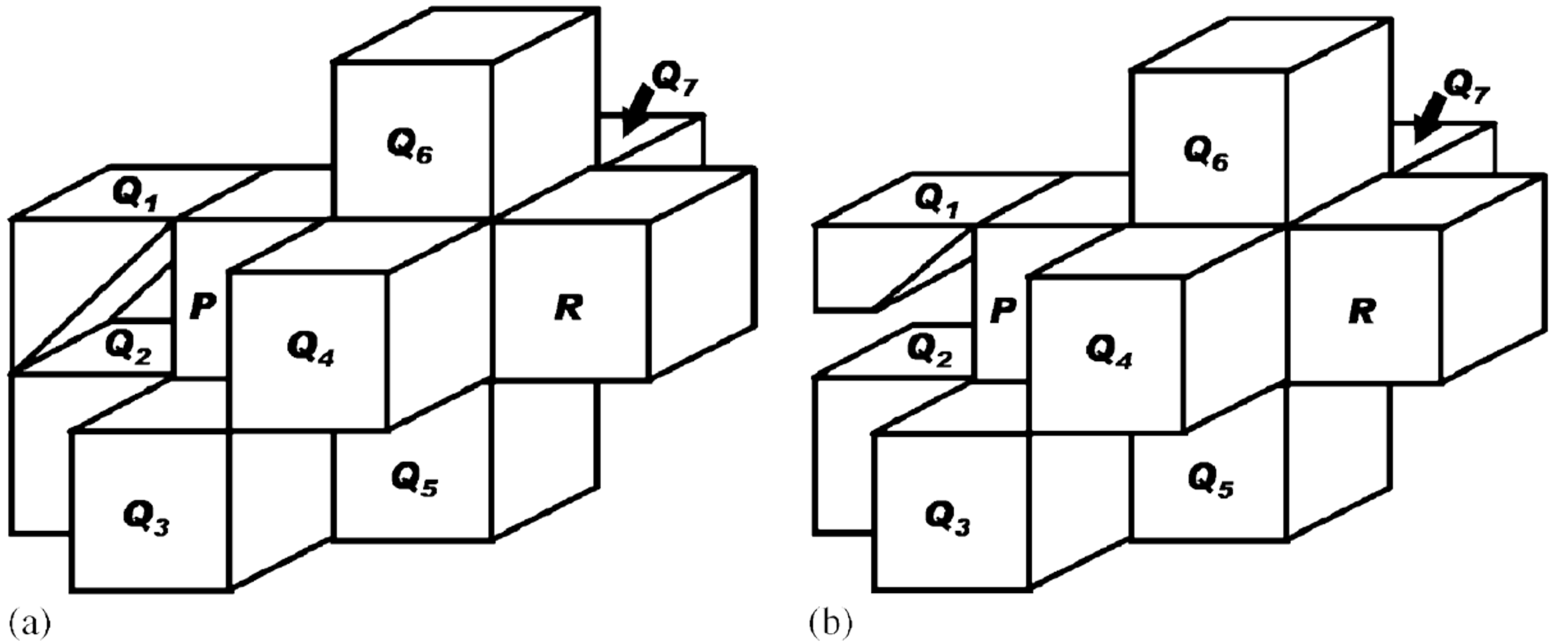

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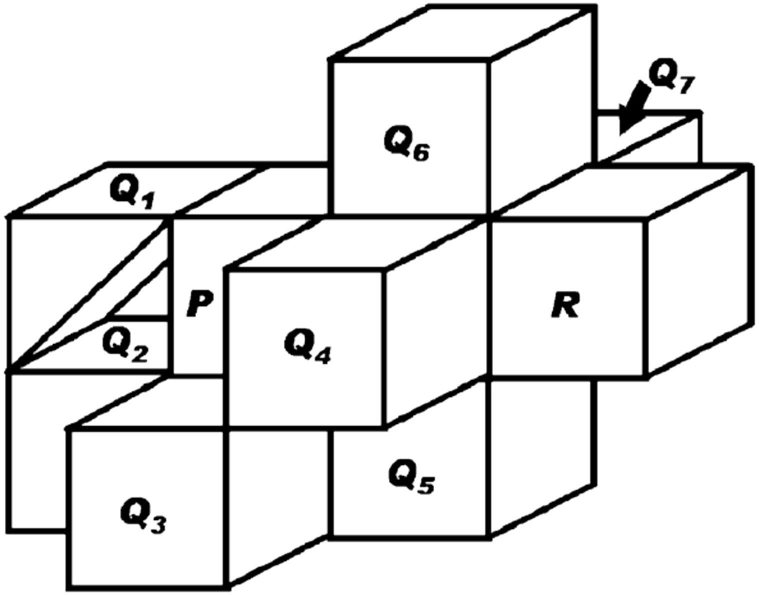

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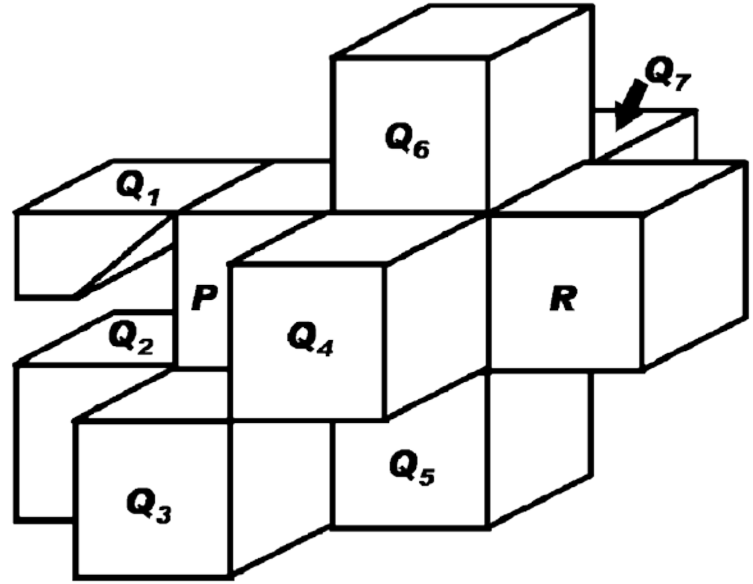

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Another example: Let $\mathcal{F} = \{F_0, ..., F_n\}$ be the set of all $(n-1)$-dimensional faces of an n-dimensional simplex. Then $\mathcal{F}$ is ***not*** strongly normal (since $F_1, ..., F_n$ are $\mathcal{F}$-neighbors of F_0, $F_1 \cap ... \cap F_n \neq \emptyset$, but $F_0 \cap F_1 \cap ... \cap F_n = \emptyset$).

When $\mathcal{F}$ is a good collection of polyhedra that is strongly normal, there is an alternative characterization of $\mathcal{F}$-homology-critical $\mathcal{F}$-$\bigcap$s:

Lemma: *Let $\mathcal{F}$ be any **strongly normal** finite set of polyhedra such that every $\mathcal{F}$-$\bigcap$ is acyclic, and let C be any $\mathcal{F}$-$\bigcap$. Then C is $\mathcal{F}$-homology-critical **just if** $\bigcup\{F \in \mathcal{F} \setminus \mathcal{F}_C \mid F$* intersects each member of $\mathcal{F}_C\}$ *is **not** acyclic.*

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So on putting C = any such Q in the above lemma we deduce:

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Cor.: *Let $\mathcal{F}$ be any **strongly normal** finite set of polyhedra such that every $\mathcal{F}$-$\bigcap$ is acyclic, and let Q be any inclusion-maximal member of $\mathcal{F}$. Then Q is $\mathcal{F}$-homology-simple **just if** $\bigcup\{F \in \mathcal{F} \setminus \{Q\} \mid F$ intersects $Q\}$ is acyclic.*

But this is ***false*** when $\mathcal{F}$ is ***not*** strongly normal:

When $\mathcal{F}$ is a good collection of polyhedra that is strongly normal, there is an alternative characterization of $\mathcal{F}$-homology-critical $\mathcal{F}$-$\bigcap$s:

Lemma: *Let $\mathcal{F}$ be any **strongly normal** finite set of polyhedra such that every $\mathcal{F}$-$\bigcap$ is acyclic, and let C be any $\mathcal{F}$-$\bigcap$. Then C is $\mathcal{F}$-homology-critical **just if** $\bigcup\{F \in \mathcal{F} \setminus \mathcal{F}_C \mid F$ intersects each member of $\mathcal{F}_C\}$ is <u>not</u> acyclic.*

Recall that, if Q is an inclusion-maximal member of $\mathcal{F}$ (so $\mathcal{F}_Q = \{Q\}$), then

Q is homology-simple in $\mathcal{F}$ $\Leftrightarrow$ Q is ***<u>not</u>*** $\mathcal{F}$-homology-critical

So on putting C = any such Q in the above lemma we deduce:

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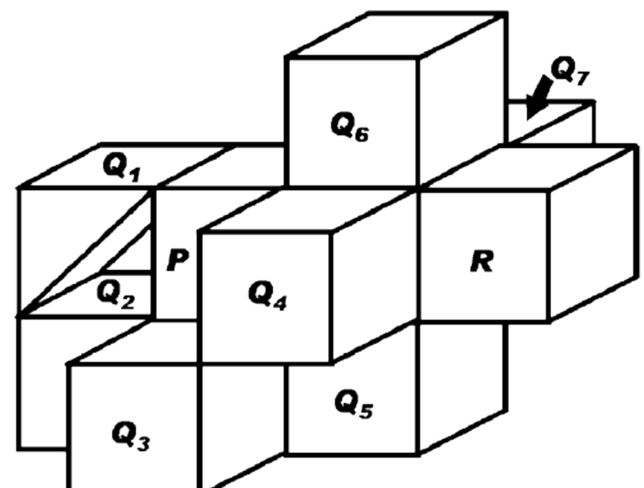
But this is ***false*** when $\mathcal{F}$ is ***not*** strongly normal:

If $\mathcal{F} = \{\mathbf{Q}_1, ..., \mathbf{Q}_7, \mathbf{P}, \mathbf{R}\}$, then
$\mathbf{Q}_1$ is ***<u>not</u>*** $\mathcal{F}$-homology-simple as
Attach($\mathbf{Q}_1$, $\mathcal{F}$) = **Core**$_{\mathcal{F}}$($\mathbf{Q}_1$) is disconnected, but
$\bigcup\{F \in \mathcal{F} \setminus \{\mathbf{Q}_1\} \mid F$ intersects $\mathbf{Q}_1\} = \mathbf{P} \cup \mathbf{Q}_2$
is acyclic.



Recall: **THEOREM:** *Let $\mathcal{F}$ be any finite set of polyhedra such that every $\mathcal{F}$-$\bigcap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then:*

(i) *Q is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$* ***if and only if*** *there is no $\mathcal{F}$-homology-critical $\mathcal{F}$-$\bigcap$ C such that $Q \in \mathcal{F}_C \subseteq \mathcal{D}$.*

(ii) *$\mathcal{D}$ is hereditarily homology-simple in $\mathcal{F}$* ***if and only if*** *there is no $\mathcal{F}$-homology-critical $\mathcal{F}$-$\bigcap$ C such that $\mathcal{F}_C \subseteq \mathcal{D}$.*

Lemma: *Let $\mathcal{F}$ be any* ***strongly normal*** *finite set of polyhedra such that every $\mathcal{F}$-$\bigcap$ is acyclic, and let C be any $\mathcal{F}$-$\bigcap$. Then C is $\mathcal{F}$-homology-critical* ***if and only if*** $\bigcup\{F \in \mathcal{F} \setminus \mathcal{F}_C \mid F$ intersects each member of $\mathcal{F}_C\}$ *is not acyclic.*

Hence:

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Following Bertrand & Couprie, we say a set $\mathcal{S}$ is an ***essential $\mathcal{F}$-clique*** if $\mathcal{S}$ has the following three properties: 1. $\mathcal{S} \subseteq \mathcal{F}$ 2. $\bigcap\mathcal{S} \neq \emptyset$ 3. $\mathcal{S} = \mathcal{F}_{\cap\mathcal{S}}$.

Readily: $\mathcal{S}$ is an essential $\mathcal{F}$-clique $\Leftrightarrow$ $\mathcal{S} = \mathcal{F}_C$ for some $\mathcal{F}$-$\bigcap$ C.

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- This result gives local characterizations of $\mathbb{P}$-homology-simpleness and hereditary-homology-simpleness in terms of the "common $\mathcal{F}$-neighbors" of essential $\mathcal{F}$-cliques (instead of cores of $\mathcal{F}$-$\bigcap$s).
- But it assumes $\mathcal{F}$ is strongly normal (unlike Main Theorems 1 & 2).
- In the case where $\mathcal{F}$ is a set of grid cells of a 3D Cartesian grid, closely related results were found by Bertrand & Couprie (2014).

Recall: **MAIN THEOREM 1** *Let $\mathcal{F}$ be a finite set of polyhedra such that every $\mathcal{F}$-$\bigcap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:*

1. *Q is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$.*
2. *Every $\mathcal{F}$-homology-critical $\mathcal{D}$-$\bigcap$ contained in Q is also contained in a member of $\mathcal{F} \setminus \mathcal{D}$.*

Proof of the 2 ⇒ 1 Part of Main Theorem 1

We say a set $\boldsymbol{C}$ of $\mathcal{F}$-$\bigcap$s is ***inclusion-closed*** if $\boldsymbol{C}$ satisfies:

- Whenever $X \in \boldsymbol{C}$ and Y is an $\mathcal{F}$-$\bigcap$ such that $Y \subseteq X$, we have that $Y \in \boldsymbol{C}$.

Lemma: *Let $\boldsymbol{C}$ be any inclusion-closed set of $\mathcal{F}$-$\bigcap$s, and let M be any inclusion-maximal member of $\boldsymbol{C}$. Then $\boldsymbol{C} \setminus \{M\}$ is an inclusion-closed set of $\mathcal{F}$-$\bigcap$s such that*:

1. $M \cap \bigcup(\boldsymbol{C} \setminus \{M\}) = \bigcup\{M \cap Z \mid Z \in \boldsymbol{C} \setminus \{M\}\}$
   $= \bigcup\{Y \mid Y \text{ is an } \mathcal{F}\text{-}\bigcap \text{ and } Y \subsetneq M\} = \mathbf{Core}_{\mathcal{F}}(M)$
2. *If M is not $\mathcal{F}$-homology-critical, then the inclusion of $\bigcup(\boldsymbol{C} \setminus \{M\})$ in $\bigcup \boldsymbol{C}$ induces a homology isomorphism.*

Assertion 2 follows from assertion 1, excision, and the exact homology sequences of $(M, M \cap \bigcup(\boldsymbol{C} \setminus \{M\}))$ and $(\bigcup \boldsymbol{C}, \bigcup(\boldsymbol{C} \setminus \{M\}))$.

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Corollary A1: *Let $\mathcal{A} \supseteq \mathcal{B}$ be any two inclusion-closed sets of $\mathcal{F}$-⋂s such that no member of $\mathcal{A} \setminus \mathcal{B}$ is $\mathcal{F}$-homology-critical. Then the inclusion of $\bigcup\mathcal{B}$ in $\bigcup\mathcal{A}$ induces a homology isomorphism.*

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Proof: Let $\mathcal{C}_0 \supseteq \ldots \supseteq \mathcal{C}_k$ be the sets of $\mathcal{F}$-⋂s defined by $\mathcal{C}_0 = \mathcal{A}$, $\mathcal{C}_k = \mathcal{B}$, and $\mathcal{C}_{j+1} = \mathcal{C}_j \setminus \{M_j\}$, M_j an inclusion-maximal member of $\mathcal{C}_j \setminus \mathcal{B}$, for $0 \leq j < k$. Here each M_j is also inclusion-maximal in $\mathcal{C}_j$, as $\mathcal{B}$ is inclusion-closed. So, for each j, the inclusion of $\bigcup \mathcal{C}_{j+1}$ in $\bigcup \mathcal{C}_j$ induces a homology isomorphism (by Lemma). Hence so does the inclusion of $\bigcup \mathcal{C}_k = \bigcup \mathcal{B}$ in $\bigcup \mathcal{C}_0 = \bigcup \mathcal{A}$. //

Recall: **Corollary A1**: *Let $\mathcal{A} \supseteq \mathcal{B}$ be any two inclusion-closed sets of $\mathcal{F}$-$\bigcap$s such that no member of $\mathcal{A} \setminus \mathcal{B}$ is $\mathcal{F}$-homology-critical. Then the inclusion of $\bigcup\mathcal{B}$ in $\bigcup\mathcal{A}$ induces a homology isomorphism.*

MAIN THEOREM 1 *Let $\mathcal{F}$ be a finite set of polyhedra such that every $\mathcal{F}$-$\bigcap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:*

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Completion of the Proof of the 2 ⇒ 1 Part of Main Theorem 1

Under the hypotheses of Main Thm. 1, let $\mathcal{S}$ be any subset of $\mathcal{D} \setminus \{Q\}$,
let $\mathcal{A}$ = set of $\mathcal{F}$-$\bigcap$s that lie in at least one member of $\mathcal{F} \setminus \mathcal{S}$
and let $\mathcal{B}$ = set of $\mathcal{F}$-$\bigcap$s that lie in at least one member of $(\mathcal{F} \setminus \mathcal{S}) \setminus \{Q\}$.
So $\mathcal{A} \setminus \mathcal{B}$ = set of $\mathcal{F}$-$\bigcap$s that lie in Q but not in any member of $(\mathcal{F} \setminus \mathcal{S}) \setminus \{Q\}$.
Now:

condition 2 of Main Thm. 1
⇒ every $\mathcal{F}$-homology-critical $\mathcal{F}$-$\bigcap$ that lies in Q also lies in a member of $\mathcal{F} \setminus \mathcal{D} \subseteq (\mathcal{F} \setminus \mathcal{S}) \setminus \{Q\}$
⇒ no member of $\mathcal{A} \setminus \mathcal{B}$ is $\mathcal{F}$-homology-critical
⇒ Q is homology-simple in $\mathcal{F} \setminus \mathcal{S}$ (by Cor. A1)
⇒ condition 1 of Main Thm. 1 (as $\mathcal{S}$ is an *arbitrary* subset of $\mathcal{D} \setminus \{Q\}$). //

Proof of the 1 ⇒ 2 Part of Main Theorem 1: A Preliminary Lemma

Recall: Whenever $\mathcal{D} \subseteq \mathcal{F}$ and C is a $\mathcal{D}$-$\bigcap$, we define $\mathcal{D}_C \overset{\text{def}}{=} \{D \in \mathcal{D} \mid C \subseteq D\}$

Lemma 5: *Suppose condition 2 is* ***not*** *satisfied. Let $Q \in \mathcal{D} \subseteq \mathcal{F}$ and let C be an $\mathcal{F}$-homology-critical $\mathcal{D}$-$\bigcap$ that is contained in Q but* ***not*** *contained in any member of $\mathcal{F} \setminus \mathcal{D}$. Then the set*

$(\bigcap \mathcal{D}_C) \cap \textbf{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C) = C \cap \textbf{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C) = C \cap \bigcup(\mathcal{F} \setminus \mathcal{D}_C)$

is the $\mathcal{F}$-core of C and is **not** *acyclic*.

Proof: As C is not contained in any member of $\mathcal{F} \setminus \mathcal{D}$, we have that

$\mathcal{D}_C = \mathcal{F}_C$ and hence $\mathcal{F} \setminus \mathcal{D}_C = \mathcal{F} \setminus \mathcal{F}_C = \{F \in \mathcal{F} \mid F \not\supseteq C\}$

Thus $C \cap \bigcup(\mathcal{F} \setminus \mathcal{D}_C) = C \cap \bigcup\{F \in \mathcal{F} \mid F \not\supseteq C\} = \textbf{Core}_{\mathcal{F}}(C)$, which is not acyclic as C is $\mathcal{F}$-homology-critical. //

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Also recall: Corollary 2: *Let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then Q is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$* ***if and only if*** **Attach**$(Q, \mathcal{F} \setminus \mathcal{S})$ *is acyclic for all $\mathcal{S} \subseteq \mathcal{D}$.*

We now prove ***not*** 2 ⇒ ***not*** 1 by showing (for $Q \in \mathcal{D} \subseteq \mathcal{F}$) that:

If $(\bigcap \mathcal{D}_C) \cap \textbf{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C)$ is ***not*** acyclic, then it is ***not*** true that "**Attach**$(Q, \mathcal{F} \setminus \mathcal{S})$ *is acyclic for all $\mathcal{S} \subseteq \mathcal{D}$*".

Proof of the 1 ⇒ 2 Part of Main Theorem 1: Properties of Acyclic Polyhedra

Property A: *Let S and T be polyhedra that satisfy any two of the following conditions. Then S and T satisfy all three conditions:*

1. *Each of S and T is acyclic*. 2. $S \cap T$ *is acyclic*. 3. $S \cup T$ *is acyclic*.

Property **A** follows from a standard result of algebraic topology—the Mayer-Vietoris exact sequence for reduced homology of polyhedra:

$$\ldots \longrightarrow \widetilde{H}_p(S \cap T) \longrightarrow \widetilde{H}_p(S) \oplus \widetilde{H}_p(T) \longrightarrow \widetilde{H}_p(S \cup T) \longrightarrow \widetilde{H}_{p-1}(S \cap T) \longrightarrow \widetilde{H}_{p-1}(S) \oplus \widetilde{H}_{p-1}(T) \longrightarrow \ldots$$

Property B: *Let* $\mathcal{P}$ *be a finite collection of polyhedra. Then the following are equivalent*:

(i) *Every nonempty subcollection of* $\mathcal{P}$ *has an acyclic intersection*:
$\bigcap \mathcal{P}'$ *is acyclic whenever* $\emptyset \neq \mathcal{P}' \subseteq \mathcal{P}$.

(ii) *Every nonempty subcollection of* $\mathcal{P}$ *has an acyclic union*:
$\bigcup \mathcal{P}'$ *is acyclic whenever* $\emptyset \neq \mathcal{P}' \subseteq \mathcal{P}$.

Property **B** follows from Property **A** by induction on the collection's size.

Recall: **MAIN THEOREM 1** *Let $\mathcal{F}$ be a finite set of polyhedra such that every $\mathcal{F}$-⋂ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:*

1. *Q is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$.*
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Notation: If C is any $\mathcal{D}$-⋂, then: $\mathcal{D}_C \overset{\text{def}}{=} \{D \in \mathcal{D} \mid C \subseteq D\}$

Lemma 5: *Suppose condition 2 is **not** satisfied. Let $Q \in \mathcal{D} \subseteq \mathcal{F}$ and let C be an $\mathcal{F}$-homology-critical $\mathcal{D}$-⋂ that is contained in Q but not contained in any member of $\mathcal{F} \setminus \mathcal{D}$. Then:*
$(\bigcap \mathcal{D}_C) \cap$ **Attach**$(Q, \mathcal{F} \setminus \mathcal{D}_C)$ *is the $\mathcal{F}$-core of C and is **not** acyclic.*

Proof of the 1 ⇒ 2 Part of Main Theorem 1

Suppose 2 is not satisfied. Then ∃ an $\mathcal{F}$-homology-critical $\mathcal{D}$-⋂, C, such that C is contained in Q but C is not contained in any member of $\mathcal{F} \setminus \mathcal{D}$.

We will deduce that Q is ***not*** $\mathbb{P}$-homology-simple for $\mathcal{D}_C$ in $\mathcal{F}$, which implies 1 is also not satisfied. To do this, we first note that:

(a) $\bigcap(\{$**Attach**$(Q,\mathcal{F}\setminus\mathcal{D}_C)\} \cup \{Q\cap D \mid D\in\mathcal{D}_C\})$

= **Attach**$(Q,\mathcal{F}\setminus\mathcal{D}_C) \cap \bigcap\mathcal{D}_C$ is ***not*** acyclic, by Lemma 5.

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Suppose 2 is not satisfied. Then ∃ an $\mathcal{F}$-homology-critical $\mathcal{D}$-$\bigcap$, C, such that C is contained in Q but C is not contained in any member of $\mathcal{F} \setminus \mathcal{D}$.

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Moreover:

(b) The $\bigcap$ of any nonempty subcollection of $\{Q \cap D \mid D \in \mathcal{D}_C\}$ is an acyclic superset of $Q \cap \bigcap \mathcal{D}_C = Q \cap C = C$.

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Recall: **Property B** *Let $\mathcal{P}$ be a finite collection of polyhedra. Then the following are equivalent*:

(i) *Every nonempty subcollection of $\mathcal{P}$ has an acyclic intersection*:

(ii) *Every nonempty subcollection of $\mathcal{P}$ has an acyclic union*:

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Suppose 2 is not satisfied. Then ∃ an $\mathcal{F}$-homology-critical $\mathcal{D}$-$\bigcap$, C, such that C is contained in Q but C is not contained in any member of $\mathcal{F} \setminus \mathcal{D}$.

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(b) and Property B imply:

(c) The $\bigcup$ of any nonempty subcollection of $\{Q\cap D \mid D\in\mathcal{D}_C\}$ is acyclic.

Proof of the 1 ⇒ 2 Part of Main Theorem 1

Suppose 2 is not satisfied. Then ∃ an $\mathcal{F}$-homology-critical $\mathcal{D}$-$\bigcap$, C, such that C is contained in Q but C is not contained in any member of $\mathcal{F} \setminus \mathcal{D}$.

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Recall: **Property B** *Let $\mathcal{P}$ be a finite collection of polyhedra. Then the following are equivalent*:

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BUT, (a) and Property B imply:

(d) ∃ a nonempty subcollection of $\{\textbf{Attach}(Q,\mathcal{F}\setminus\mathcal{D}_C)\} \cup \{Q\cap D \mid D\in\mathcal{D}_C\}$ whose ⋃ is ***not*** acyclic.

Proof of the 1 ⇒ 2 Part of Main Theorem 1

Suppose 2 is not satisfied. Then ∃ an $\mathcal{F}$-homology-critical $\mathcal{D}$-⋂, C, such that C is contained in Q but C is not contained in any member of $\mathcal{F} \setminus \mathcal{D}$.

We will deduce that Q is ***not*** $\mathbb{P}$-homology-simple for $\mathcal{D}_C$ in $\mathcal{F}$, which implies 1 is also not satisfied.

(a), (b), and Property B imply:

(c) The ⋃ of any nonempty subcollection of $\{Q \cap D \mid D \in \mathcal{D}_C\}$ is acyclic.

(d) ∃ a nonempty subcollection of $\{\mathbf{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C)\} \cup \{Q \cap D \mid D \in \mathcal{D}_C\}$ whose ⋃ is ***not*** acyclic.

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Suppose 2 is not satisfied. Then ∃ an $\mathcal{F}$-homology-critical $\mathcal{D}$-$\bigcap$, C, such that C is contained in Q but C is not contained in any member of $\mathcal{F} \setminus \mathcal{D}$.

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(c) and (d) imply:

(e) $\exists\, \mathcal{T} \subseteq \mathcal{D}_C : \mathbf{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C) \cup \bigcup\{Q \cap D \mid D \in \mathcal{T}\}$ is ***not*** acyclic.

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Proof of the 1 ⇒ 2 Part of Main Theorem 1

Suppose 2 is not satisfied. Then ∃ an $\mathcal{F}$-homology-critical $\mathcal{D}$-$\bigcap$, C, such that C is contained in Q but C is not contained in any member of $\mathcal{F} \setminus \mathcal{D}$. We will deduce that Q is ***not*** $\mathbb{P}$-homology-simple for $\mathcal{D}_C$ in $\mathcal{F}$, which implies 1 is also not satisfied.

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Proof of the 1 ⇒ 2 Part of Main Theorem 1

Suppose 2 is not satisfied. Then ∃ an $\mathcal{F}$-homology-critical $\mathcal{D}$-$\bigcap$, C, such that C is contained in Q but C is not contained in any member of $\mathcal{F} \setminus \mathcal{D}$. We will deduce that Q is ***not*** $\mathbb{P}$-homology-simple for $\mathcal{D}_C$ in $\mathcal{F}$, which implies 1 is also not satisfied.

(a), (b), and Property B imply:

(c) The $\bigcup$ of any nonempty subcollection of $\{Q \cap D \mid D \in \mathcal{D}_C\}$ is acyclic.

(d) ∃ a nonempty subcollection of $\{\mathbf{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C)\} \cup \{Q \cap D \mid D \in \mathcal{D}_C\}$ whose $\bigcup$ is ***not*** acyclic.

(c) and (d) imply:

(e) $\exists\, \mathcal{T} \subseteq \mathcal{D}_C : \mathbf{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C) \cup \bigcup\{Q \cap D \mid D \in \mathcal{T}\}$ is ***not*** acyclic.

(f) For all $\mathcal{T} \subseteq \mathcal{D}_C$ we have that:

$$\mathbf{Attach}(Q, \mathcal{F} \setminus \mathcal{D}_C) \cup \bigcup\{Q \cap D \mid D \in \mathcal{T}\} = Q \text{ or } \mathbf{Attach}(Q, \mathcal{F} \setminus (\mathcal{D}_C \setminus \mathcal{T}))$$

according to whether $Q \in \mathcal{T}$ or $Q \notin \mathcal{T}$.

As Q is acyclic, (e) and (f) imply:

$\exists\, \mathcal{T} \subseteq \mathcal{D}_C : \mathbf{Attach}(Q, \mathcal{F} \setminus (\mathcal{D}_C \setminus \mathcal{T}))$ is ***not*** acyclic.

Proof of the 1 ⇒ 2 Part of Main Theorem 1

As Q **is acyclic, (e) and (f) imply:**

$\exists\, \mathcal{T} \subseteq \mathcal{D}_C$: **Attach**$(Q, \mathcal{F} \setminus (\mathcal{D}_C \setminus \mathcal{T}))$ is ***not*** acyclic.

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Proof of the 1 ⇒ 2 Part of Main Theorem 1

As Q is acyclic, (e) and (f) imply:

$\exists\ \mathcal{T} \subseteq \mathcal{D}_C$: **Attach**$(Q, \mathcal{F} \setminus (\mathcal{D}_C \setminus \mathcal{T}))$ is ***not*** acyclic.

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Equivalently:

$\exists\ \mathcal{S} \subseteq \mathcal{D}_C$: **Attach**$(Q, \mathcal{F} \setminus \mathcal{S})$ is ***not*** acyclic.

Equivalently (by Corollary 2):

Q is ***not*** $\mathbb{P}$-homology-simple for $\mathcal{D}_C$ in $\mathcal{F}$.

So we have shown that 1 is not satisfied. This completes the proof. //

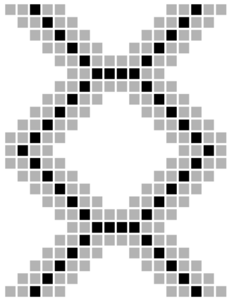
Recall: **Corollary 2**: *Let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then Q is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$* ***if and only if*** **Attach**$(Q, \mathcal{F} \setminus \mathcal{S})$ *is acyclic for all $\mathcal{S} \subseteq \mathcal{D}$.*

MAIN THEOREM 1 *Let $\mathcal{F}$ be a finite set of polyhedra such that every $\mathcal{F}$-$\bigcap$ is acyclic, and let $Q \in \mathcal{D} \subseteq \mathcal{F}$. Then the following are equivalent:*

1. *Q is $\mathbb{P}$-homology-simple for $\mathcal{D}$ in $\mathcal{F}$.*
2. *Every $\mathcal{F}$-homology-critical $\mathcal{D}$-$\bigcap$ contained in Q is also contained in a member of $\mathcal{F} \setminus \mathcal{D}$.*

Summary (1)

- A ***thinning algorithm*** simplifies a binary image by reducing its foreground to a thin "skeleton" in a "topology-preserving" way.



- Bertrand's *critical kernels* have been studied extensively by Bertrand and Couprie, who have used them to design many parallel thinning algorithms that *automatically* satisfy the requirement of being topology-preserving.

- This talk has presented a variant of the concept of *critical kernels*: ***homology-critical*** kernels. For sets of grid cells of a 2D, 3D, or 4D Cartesian grid, *homology-critical* and *critical* are equivalent.

- Many results about critical kernels of such sets become valid for sets of *arbitrary convex polytopes of any dimension* (and, more generally, sets of *arbitrary acyclic polyhedra whose nonempty intersections are acyclic*) if they are restated as results about homology-critical kernels.

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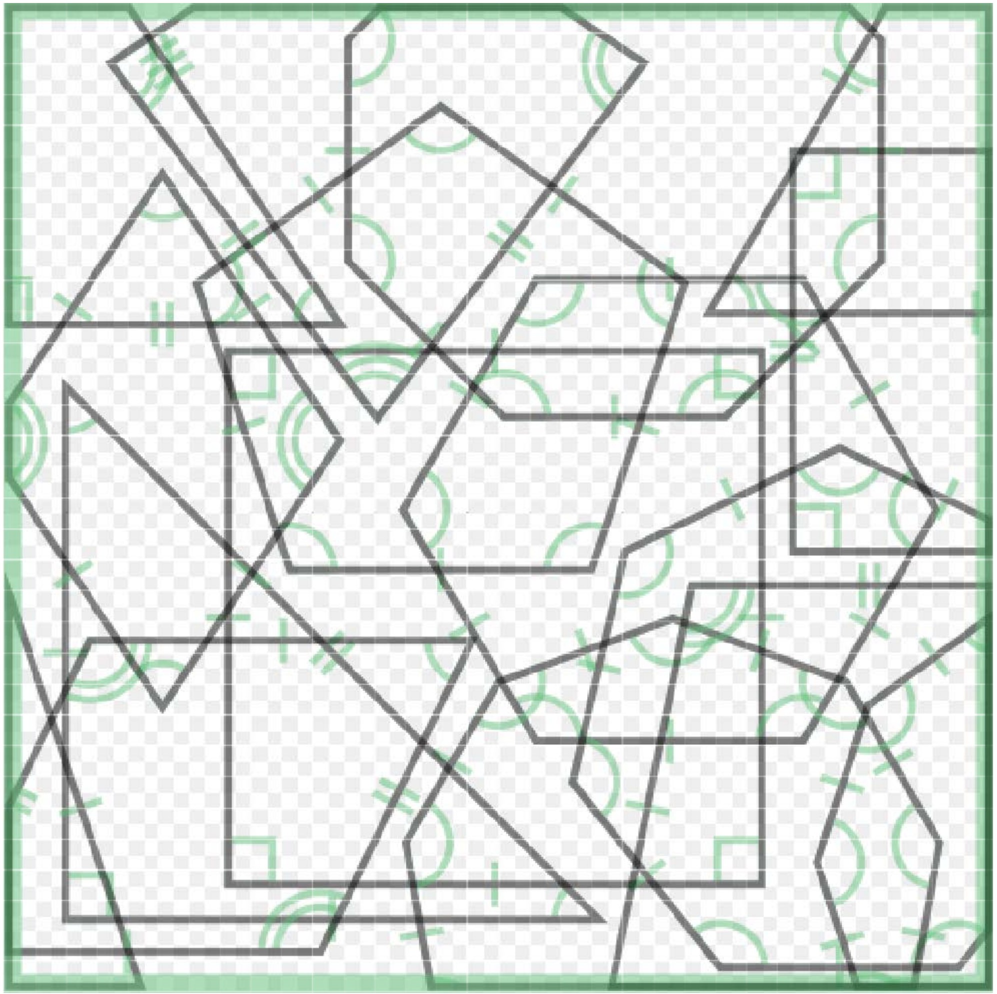
A 2D example of a collection of polyhedra to which the main results of this talk would apply:

- The polyhedra here are the 2D convex polytopes bounded by the gray lines.
- The green parts of this drawing are irrelevant.



Summary (2)

- One formulation of the requirement that a thinning algorithm be "topology-preserving" is that the set of deleted image elements satisfy the condition of being ***homology-simple*** in the image foreground $\mathcal{F}$.
- For binary images on *grid cells of a* 2*D*, 3*D*, *or* 4*D Cartesian grid*, a fundamental theorem of Bertrand & Couprie (2009) relating to critical kernels provides a useful local necessary and sufficient condition for all subsets of a given set of image elements to be homology-simple in $\mathcal{F}$.
- Main Theorem 2 substitutes *homology-critical* for *critical* in the Bertrand-Couprie theorem, to give an analogous necessary and sufficient condition that is valid for binary images on sets of *arbitrary convex polytopes of any dimension* (even if some polytopes have overlapping interiors) and, more generally, *arbitrary acyclic polyhedra whose nonempty intersections are acyclic*.
- When $\mathcal{F}$ is a set of 3D Cartesian grid cells, Bertrand & Couprie (2014) established that their results can be stated in terms of the *common neighbors of essential cliques* (instead of cores of $\mathcal{F}$-intersections). This is also true of our main results if $\mathcal{F}$ is ***strongly normal*** (2-Helly).