Decomposition and Construction of Higher-dimensional Neighbourhood Operations

Atsushi Imiya

Institute of Management and Information Technologies, Chiba University
Yayoi-cho 1-33, Inage-ku, Chiba 263-8522, Japan

Abstract. We introduce a method to construct morphological operations in a higher-dimensional digital space from a collection of set operations along digital isothetic lines in a space. First, we prove that the $2n$-neighbourhood in an n-dimensional digital space is decomposed into the $2(n-1)$-neighbourhoods in the mutually orthogonal $(n-1)$-dimensional digital spaces. Second, we derive a method to construct the object boundary in an n-dimensional digital space form the digital boundaries in the mutually orthogonal $(n-1)$-dimensional digital spaces. This decomposition and construction relation of the neighbourhoods and boundaries implies that the object boundary in an n-dimensional digital space can be computed as the union of the endpoints of isothetic digital lines intersecting with the digital object in the digital space.

1 Introduction

We develop a method to construct higher-dimensional digital morphological operations from a collection of one-dimensional set operations along digital isothetic lines in a space. This decomposition property allows us to construct the neighbourhood operations in a fine digital space from these in coarse digital spaces [1]. This construction method of the neighbourhood operations allows us to define and compute the boundary of digital objects in a higher-dimensional digital space [2] using set operations on digital isothetic lines.

2 Mathematical Preliminaries

Setting $\mathbf{R}^n$ to be an n-dimensional Euclidean space, we express vectors in $\mathbf{R}^n$ as $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^\top$. Let $\mathbf{Z}$ be the set of all integers. The n-dimensional digital space $\mathbf{Z}^n$ is set of all $\boldsymbol{x}$ for which all x_i are integers.

Definition 1. *The voxels centred at the point $\boldsymbol{y} \in \mathbf{Z}^n$ in $\mathbf{R}^n$ is*

$$\mathbf{V}(\boldsymbol{y}) = \left\{ \boldsymbol{x} \,\middle|\, |\boldsymbol{x} - \boldsymbol{y}|_\infty \leq \frac{1}{2} \right\}. \tag{1}$$

In this paper, we deal with the connectivity and adjacency of the centroids of the voxels [3], which are elements of $\mathbf{Z}^n$.

The sets $\mathbf{F} \oplus \mathbf{G}$ and $\mathbf{F} \ominus \mathbf{G}$ such that

$$\mathbf{F} \oplus \mathbf{G} = \bigcup_{\boldsymbol{y} \in \mathbf{G}} \left(\bigcup_{\boldsymbol{x} \in \mathbf{F}} \{\boldsymbol{x} + \boldsymbol{y}\right), \qquad \mathbf{F} \ominus \mathbf{G} = \bigcap_{\boldsymbol{y} \in \mathbf{G}} \left(\bigcup_{\boldsymbol{x} \in \mathbf{F}} \{\boldsymbol{x} + \boldsymbol{y}\} \right) \tag{2}$$

are called the Minkowski addition and Minkowski subtraction [4] of $\mathbf{F}$ and $\mathbf{G}$, respectively. The translation of $\mathbf{F}$ by $\boldsymbol{a} \in \mathbf{Z}^n$ is

$$\mathbf{F}(\boldsymbol{a}) = \bigcup_{\boldsymbol{x} \in \boldsymbol{F}} \{\boldsymbol{x} + \boldsymbol{a}\} = \mathbf{F} \oplus \{\boldsymbol{a}\}. \tag{3}$$

For the Minkowski addition and subtraction, the relations

$$\mathbf{F} \ominus \mathbf{G} = \overline{\overline{\mathbf{F}} \oplus \overline{\mathbf{G}}}, \tag{4}$$
$$\mathbf{F} \oplus (\mathbf{G} \cup \mathbf{H}) = (\mathbf{F} \oplus \mathbf{G}) \cup (\mathbf{F} \oplus \mathbf{H}), \tag{5}$$
$$\mathbf{F} \ominus (\mathbf{G} \cup \mathbf{H}) = (\mathbf{F} \ominus \mathbf{G}) \cap (\mathbf{F} \ominus \mathbf{H}) \tag{6}$$

are satisfied. Furthermore, we obtain the following lemma.

Lemma 1. *If* $\mathbf{F} \cap \mathbf{G} = \emptyset$, *the equalities*

$$(\mathbf{F} \cup \mathbf{G}) \oplus \mathbf{H} = (\mathbf{F} \oplus \mathbf{H}) \cup (\mathbf{G} \oplus \mathbf{H}), \tag{7}$$
$$(\mathbf{F} \cup \mathbf{G}) \ominus \mathbf{H} = (\mathbf{F} \ominus \mathbf{H}) \cup (\mathbf{G} \ominus \mathbf{H}) \tag{8}$$

are satisfied.

(*Proof*)

$$\begin{aligned}
(\mathbf{F} \cup \mathbf{G}) \oplus \mathbf{H} &= \bigcup_{\boldsymbol{x} \in \mathbf{H}} (\mathbf{F} \cup \mathbf{G})(\boldsymbol{x}) \\
&= \{\boldsymbol{x} + \boldsymbol{y} | \forall \boldsymbol{x} \in \mathbf{H}, \forall \boldsymbol{y} \in (\mathbf{F} \cup \mathbf{G})\} \\
&= \{\boldsymbol{x} + \boldsymbol{y} | \forall \boldsymbol{x} \in \mathbf{H}, \forall \boldsymbol{y} \in \mathbf{F}\} \cup \{\boldsymbol{x} + \boldsymbol{y} | \forall \boldsymbol{x} \in \mathbf{H}, \forall \boldsymbol{y} \in \mathbf{G}\} \\
&= (\mathbf{F} \oplus \mathbf{H}) \cup (\mathbf{G} \oplus \mathbf{H}), \\
(\mathbf{F} \cup \mathbf{G}) \ominus \mathbf{H} &= \bigcap_{\boldsymbol{x} \in \mathbf{H}} (\mathbf{F} \cup \mathbf{G})(\boldsymbol{x}) \\
&= \{\boldsymbol{x} + \boldsymbol{y} | \forall \boldsymbol{x} \in \mathbf{H}, \exists \boldsymbol{y} \in (\mathbf{F} \cup \mathbf{G})\} \\
&= \{\boldsymbol{x} + \boldsymbol{y} | \forall \boldsymbol{x} \in \mathbf{H}, \exists \boldsymbol{y} \in \mathbf{F}\} \cup \{\boldsymbol{x} + \boldsymbol{y} | \forall \boldsymbol{x} \in \mathbf{H}, \exists \boldsymbol{y} \in \mathbf{G}\} \\
&= (\mathbf{F} \ominus \mathbf{H}) \cup (\mathbf{G} \ominus \mathbf{H}).
\end{aligned}$$

□

For $\mathbf{F} \subset \mathbf{Z}^n$, we define

$$\mathbf{F}_k = \{\boldsymbol{x} \,|\, \boldsymbol{x} \in \mathbf{F}, x_k = 0\} \tag{9}$$

and

$$\mathbf{F}_{k\alpha} = \{\boldsymbol{x} \mid \boldsymbol{x} \in \mathbf{F}_k \oplus \{\alpha \boldsymbol{e}_k\}, \alpha \in \mathbf{Z}\}. \tag{10}$$

For

$$\alpha_+(k) = \max_{\mathbf{F}\cap\mathbf{F}_{k\alpha}\neq\emptyset} \alpha, \qquad \alpha_-(k) = \min_{\mathbf{F}\cap\mathbf{F}_{k\alpha}\neq\emptyset} \alpha, \tag{11}$$

setting

$$\mathcal{N}(k) = \{\alpha | \alpha_-(k) \leq \alpha \leq \alpha_+(k)\}, \tag{12}$$

$\mathbf{F}_{k\alpha}$ satisfies the relation

$$\mathbf{F} = \bigcup_{k=1}^{n} \left(\bigcup_{\alpha\in\mathcal{N}(k)} \mathbf{F}_{k\alpha} \right). \tag{13}$$

Equation (13) is the multidirectional multislice decomposition of a digital point set. Furthermore, the relations

$$\mathbf{F}_{k\alpha} = \bigcup_{l=1}^{n-1} \left(\bigcup_{\beta\in\mathcal{N}(l)} \mathbf{F}_{k\alpha\, l\beta} \right), \tag{14}$$

$$\mathbf{F}_{k\alpha\, l\beta}, = \bigcup_{m=1}^{n-2} \left(\bigcup_{\gamma\in\mathcal{N}(m)} \mathbf{F}_{k\alpha\, l\beta\, m\gamma} \right) \tag{15}$$

are satisfied. Equations (13), (14) and (15) derive the hierarchical decomposition of the point sets in the form

$$\mathbf{F}_{k(l)\alpha(l)} = \bigcup_{k(l)=1}^{n-l} \left(\bigcup_{\alpha(l)\in\mathcal{N}(k(l))} \mathbf{F}_{k(l)\alpha(l)} \right) \tag{16}$$

for $l = 0, 1, 2, \cdots, n-1$. Figure 1 shows the multidirectional multislice decomposition of a digital point set in a three-dimensional digital space.

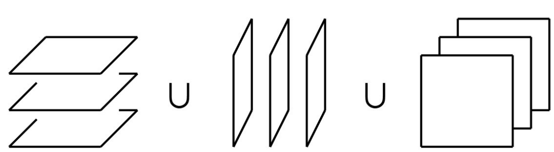


Fig. 1. Digital point set and its decomposition. The multidirectional multislice decomposition of a digital point set in a three-dimensional digital space is shown.

3 Neighbourhood Operations

The $2n$-neighbourhood of the origin in $\mathbf{Z}^n$ is

$$\mathbf{N}^n = \{\boldsymbol{x} \,|\, |x_i| = 1,\ \boldsymbol{x} = (x_1, x_2, \cdots, x_n)^\top\}. \tag{17}$$

Let $\mathbf{N}(\boldsymbol{x}) = \mathbf{N} \oplus \{\boldsymbol{x}\}$ for $\boldsymbol{x} \in \mathbf{Z}^n$.

Definition 2. *If $\boldsymbol{y} \in \mathbf{N}(\boldsymbol{x})$ and $\boldsymbol{x} \in \mathbf{N}(\boldsymbol{y})$, $\boldsymbol{x}$ and $\boldsymbol{y}$ are connected to each other.*

Definition 3. *For $\boldsymbol{y} \notin \mathbf{N}(\boldsymbol{x})$, if there exists at least one sequence $\boldsymbol{p}_{i+1} \in \mathbf{N}(\boldsymbol{p}_i)$ and $\boldsymbol{p}_i \in \mathbf{N}(\boldsymbol{p}_{i+1})$ for $i = 1, 2, \cdots k-1$, the string $\{\boldsymbol{p}\}_{i=1}^k$ is a path from $\boldsymbol{p}_1 := \boldsymbol{x}$ to $\boldsymbol{p}_k := \boldsymbol{y}$.*

Definition 4. *For a pair of points $\boldsymbol{x}$ and $\boldsymbol{y}$, if there exists a path between them, this pair is connected.*

Definition 5. *For $\mathbf{F} \in \mathbf{Z}^n$, if there exist at least a path between any pairs of points in $\mathbf{F}$, $\mathbf{F}$ is a connected component.*

On the digital line $\mathbf{Z}$, the neighbourhood $\mathbf{N}^1$ of the point 0 is $\mathbf{N}^1 = \{-1, 1\}$ and a digital object is a string of points $\mathbf{O} = \{k\}_{k=n}^m$ for $m > n$ and $m, n \in \mathbf{Z}$. The Minkowski addition and subtraction for a pair of sets $\mathbf{A}$ and $\mathbf{B}$ on the digital line $\mathbf{Z}$ are

$$\mathbf{A} \oplus \mathbf{B} = \{\cup(a+b) \,|\, a \in \mathbf{A}, b \in \mathbf{B}\}, \quad \mathbf{A} \ominus \mathbf{B} = \{\cup(a-b) \,|\, a \in \mathbf{A}, b \in \mathbf{B}\}. \tag{18}$$

Example 1. The dilation and erosion of a collection of points are concatenation and elimination of points to both endpoints of a string, respectively, such that

$$\mathbf{O} \oplus \mathbf{N}^1 = \{k\}_{n-1}^{m+1}, \qquad \mathbf{O} \ominus \mathbf{N}^1 = \{k\}_{n+1}^{m-1}, \tag{19}$$

assuming $(m-1) + (n+1) \geq 0$.

From the linear neighbourhood in $\mathbf{Z}^n$ such that

$$\mathbf{N}_k^1 = \{\boldsymbol{x} \,|\, |x_k| = 1,\ x_i = 0,\ i \neq k\}, \tag{20}$$

we can construct $\mathbf{N}^n$ as

$$\mathbf{N}^n = \bigcup_{k=1}^n \mathbf{N}_k^1, \tag{21}$$

$$\mathbf{N}^n = \bigcup_{k=1}^n \mathbf{N}_k^{n-1}, \qquad \mathbf{N}_k^{n-1} = \mathbf{N}^n \setminus \mathbf{N}_k^1, \tag{22}$$

$$\mathbf{N}_k^{n-1} = \bigcup_{l=1}^{n-1} \mathbf{N}_{kl}^{n-2}, \qquad \mathbf{N}_{kl}^{n-2} = \mathbf{N}_k^{n-1} \setminus \mathbf{N}_l^1, \tag{23}$$

$$\mathbf{N}_{kl}^{n-2} = \bigcup_{m=1}^{n-2} \mathbf{N}_{klm}^{n-3}, \qquad \mathbf{N}_{klm}^{n-3} = \mathbf{N}_{kl}^{n-2} \setminus \mathbf{N}_m^1. \tag{24}$$

Equations (22), (23) and (24) imply that a neighbourhood in a higher-dimensional digital space can be decomposed into the union of neighbourhoods in lower-dimensional digital spaces. This recursive relation is expressed as

$$\mathbf{N}^{n-l}_{k(1)k(2)\cdots k(l)} = \bigcup_{k(l)=1}^{n-l} \mathbf{N}^{n-(l+1)}_{k(1)k(2)\cdots k(l+1)},$$
$$\mathbf{N}^{n-(l+1)}_{k(1)k(2)\cdots k(l+1)} = \mathbf{N}^{n-l}_{k(1)k(2)\cdots k(l)} \setminus \mathbf{N}^{1}_{k(l+1)}, \tag{25}$$

for $l = 0, 1, 2, \cdots n-1$. Figure 2 shows that the 8-neighbourhood in a four-dimensional digital space is decomposed into four mutually orthogonal 6-neighbourhoods in the three-dimensional digital spaces.

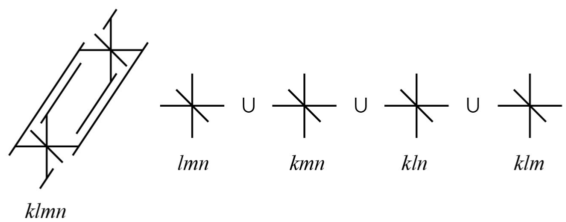


Fig. 2. Decomposition of a neighbourhood. The 8-neighbourhood in a four-dimensional digital space is decomposed into four mutually orthogonal 6-neighbourhoods in the three-dimensional digital spaces.

Figure 3 illustrates the neighbourhood operations on the horizontal and vertical isothetic lines on a digital plane. The connectivity of the four connected object shown in (a) is computed by using the connectivity on the horizontal and the vertical isothetic lines on the digital plane.

4 Digital Complex and Digital Object

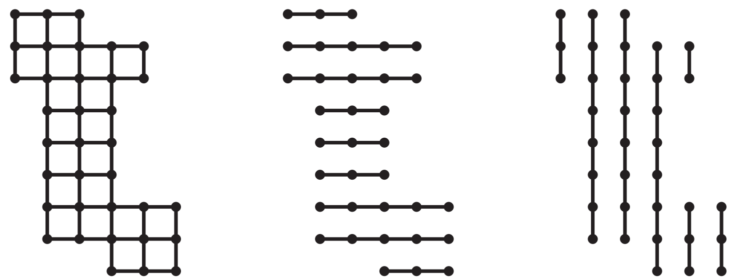
Let $\boldsymbol{e}_k$ be the unit vector whose kth element is 1. The digital n-simplex with $2n$-connectivity in $\mathbf{Z}^n$ is

$$\mathbf{S} = \left\{ \boldsymbol{v}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \,\middle|\, \boldsymbol{v}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) = \sum_{k=1}^{n} \varepsilon_k \boldsymbol{e}_i, \ \ \varepsilon_i \in \{0, 1\} \right\}. \tag{26}$$

We define the digital n-complex using $\mathbf{S}$.

Definition 6. *The digital n-complex is a union of connected simplices.*

Definition 7. *The digital thick n-complex is a union of simplices connected by $(n-1)$-simplices.*



(a) (b) (c)

Fig. 3. One-dimensional operations for a two-dimensional object. (a) Four-connected object on the digital plane. (b) Neighbourhood operations on the horizontal isothetic lines on the digital plane. (c) Neighbourhood operations on the vertical isothetic lines on the digital plane.

Using digital thick n-complices, we define a digital object.

Definition 8. *If the number of connected simplices in a thick n-complex $\mathbf{F}$ is finite and if the complement of $\mathbf{F}$ is a thick n-complex, we call $\mathbf{F}$ a digital object.*

These definitions imply that an object contains k-simplices for $k \leq n-1$ as the connected components. If $n = 3$, 1- and 2-simplices are called digital needle and walls, respectively.

Definition 9. *We call a connected component of k-simplices for $k \leq (n-1)$ a thin object.*

The minimum thickness of a thin object is one.

Example 2. On $\mathbf{Z}$, a digital object is a finite union of finite intervals

$$\mathbf{I} = \bigcup_{i=1}^{n} \mathbf{I}_i, \qquad \mathbf{I}_i = [a_i, b_i] \tag{27}$$

for $a_i < a_{i+1}$ and $b_i < b_{i+1}$ with the condition $(a_{i+1} - b_i) \geq 3$.

Figure 4 shows a digital object in a one-dimensional digital space. From top to bottom, a point set, its Euclidean embedding and the embedding of the dual set are shown.

For an object $\mathbf{F} \in \mathbf{Z}^n$, the embedding of $\mathbf{F}$ into $\mathbf{R}^n$ is

$$\mathcal{F} = \bigcup_{\boldsymbol{x} \in \mathbf{F}} \mathbf{V}(\boldsymbol{x}). \tag{28}$$

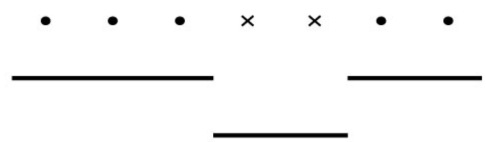


Fig. 4. Operations on a digital line. From top to bottom, a point set, its Euclidean embedding and the embedding of the dual set are shwon.

The polytope $\mathcal{F}$ is an isothetic Nef-polytope [5], which is a union of voxels connected by the faces of voxels. The vertices of $\mathcal{F}$ lie on the dual grid

$$\mathbf{D}^n = \mathbf{Z}^n + \left\{ \frac{1}{2}\boldsymbol{e} \right\}, \qquad \boldsymbol{e} = \sum_{i=1}^{n} \boldsymbol{e}_i \tag{29}$$

of $\mathbf{Z}^n$.

5 Digital Boundary Manifold

We define the boundary of a point set in $\mathbf{Z}^n$.

Definition 10. *For a point set* $\mathbf{F}$*, we call*

$$\partial_- \mathbf{F} = \mathbf{F} \setminus (\mathbf{F} \ominus \mathbf{N}^n), \qquad \partial_+ \mathbf{F} = (\mathbf{F} \oplus \mathbf{N}^n) \setminus \mathbf{F} \tag{30}$$

the internal and external boundaries of $\mathbf{F}$*, respectively.*

For the internal and external boundaries, we have the following relations

$$\mathbf{F} \setminus (\mathbf{F} \ominus \mathbf{N}^n) = \bigcup_{k=1}^{n} \bigcup_{\alpha \in \mathcal{N}(k)} \left(\mathbf{F}_{k\alpha} \setminus (\mathbf{F}_{k\alpha} \ominus \mathbf{N}_k^{n-1}) \right), \tag{31}$$

$$(\mathbf{F} \oplus \mathbf{N}^n) \setminus \mathbf{F} = \bigcup_{k=1}^{n} \bigcup_{\alpha \in \mathcal{N}(k)} \left((\mathbf{F}_{k\alpha} \oplus \mathbf{N}_k^{n-1}) \setminus \mathbf{F}_{k\alpha} \right). \tag{32}$$

These properties imply the following theorem.

Theorem 1. *The boundary* $\partial_\pm \mathbf{F}$ *of an* n*-dimensional digital object* $\mathbf{F}$ *is the union of its* $(n-1)$*-dimensional boundaries.*

Therefore, it is possible to construct $\partial_{\pm}\mathbf{F}$ from $\partial_{\pm}\mathbf{F}_{k\alpha}$. Furthermore, eqs. (14), (22) and (23) imply the relations

$$\mathbf{F}_{k\alpha} \setminus (\mathbf{F}_{k\alpha} \ominus \mathbf{N}_k^{n-1}) = \bigcup_{l=1}^{n-1} \bigcup_{\beta\in\mathcal{N}(l)} \left(\mathbf{F}_{k\alpha\, l\beta} \setminus (\mathbf{F}_{k\alpha\, l\beta} \ominus \mathbf{N}_{kl}^{n-2})\right), \tag{33}$$

$$(\mathbf{F}_{k\alpha} \oplus \mathbf{N}_k^{n-1}) \setminus \mathbf{F}_{k\alpha} = \bigcup_{l=1}^{n} \bigcup_{\beta\in\mathcal{N}(l)} \left((\mathbf{F}_{k\alpha\, l\beta} \oplus \mathbf{N}_{kl}^{n-2}) \setminus \mathbf{F}_{k\alpha\, l\beta}\right). \tag{34}$$

Decomposing both a digital object and its neighbourhood by using eqs. (16) and (25), respectively, we have the recursive forms

$$\begin{aligned}&\mathbf{F}_{k(l)\alpha(l)} \setminus (\mathbf{F}_{k(l)\alpha(l)} \ominus \mathbf{N}^{n-l}_{k(1)k(2)\cdots k(l)})\\ &= \bigcup_{k(l+1)=1}^{n-l} \bigcup_{\alpha(l+1)\in\mathcal{N}(k(l+1))} \Big(\mathbf{F}_{k(l+1)\alpha(l+1)} \setminus (\mathbf{F}_{k(l+1)\alpha(l+1)} \ominus \mathbf{N}^{n-l}_{k(1)k(2)\cdots k(l+1)})\Big),\end{aligned} \tag{35}$$

$$\begin{aligned}&(\mathbf{F}_{k(l)\alpha(l)} \oplus \mathbf{N}^{n-l}_{k(1)k(2)\cdots k(l)}) \setminus \mathbf{F}_{k(l)\alpha(l)}\\ &= \bigcup_{k(l+1)=1}^{n-l} \bigcup_{\alpha(l+1)\in\mathcal{N}(k(l+1))} \Big((\mathbf{F}_{k(l+1)\alpha(l+1)} \oplus \mathbf{N}^{n-l}_{k(1)k(2)\cdots k(l+1)}) \setminus \mathbf{F}_{k(l+1)\alpha(l+1)}\Big)\end{aligned} \tag{36}$$

for the internal and external digital boundaries, where $l = 0, 1, 2, \cdots, n-2$. Using these relations recursively, we can construct the boundary-detection algorithm for n-dimensional digital objects from one-dimensional boundary detection algorithms.

Let

$$\mathbf{l}(k, \boldsymbol{\alpha}_k) = \{\boldsymbol{x} = t\boldsymbol{e}_k + \sum_{i\neq k} \alpha_i \boldsymbol{e}_i,\ \ \alpha_i \in \mathbf{Z}\}, \tag{37}$$

where $\boldsymbol{\alpha}_k = \{\alpha_i\}_{i=1,\, i\neq k}^n$, and

$$\mathbf{F}_{k,\boldsymbol{\alpha}} = \mathbf{F} \bigcap \mathbf{l}(k, \boldsymbol{\alpha}_k). \tag{38}$$

Then, by computing the one-dimensional internal and external boundaries

$$\partial_-\mathbf{F}_{k,\boldsymbol{\alpha}} = \mathbf{F}_{k,\boldsymbol{\alpha}_k} \setminus (\mathbf{F}_{k,\boldsymbol{\alpha}_k} \ominus \mathbf{N}^1), \tag{39}$$

$$\partial_+\mathbf{F}_{k,\boldsymbol{\alpha}_k} = (\mathbf{F}_{k,\boldsymbol{\alpha}_k} \oplus \mathbf{N}^1) \setminus \mathbf{F}_{k,\boldsymbol{\alpha}_k} \tag{40}$$

for all $\{\boldsymbol{e}_k, \boldsymbol{\alpha}_k\}_{k=1}^n$, the n-dimensional internal and external boundaries, respectively, are constructed as

$$\partial_-\mathbf{F} = \bigcup_k \bigcup_{\boldsymbol{\alpha}_k} \partial_-\mathbf{F}_{k,\boldsymbol{\alpha}_k}, \tag{41}$$

$$\partial_+\mathbf{F} = \bigcup_k \bigcup_{\boldsymbol{\alpha}_k} \partial_+\mathbf{F}_{k,\boldsymbol{\alpha}_k}. \tag{42}$$

Example 3. For $\mathbf{I}$ *in eq. (27), we have the expression*

$$\mathbf{I} = \bigcup_{i=1}^{n} \left\{ a_i - \frac{1}{2}, b_i + \frac{1}{2} \right\}. \tag{43}$$

Since

$$\left(a_{i+1} - \frac{1}{2} \right) - \left(b_i + \frac{1}{2} \right) = (a_{i+1} - b_i) - \frac{1}{2} - \frac{1}{2} \geq 2, \tag{44}$$

on the background between a pair of successive intervals $\mathbf{I}_i$ and $\mathbf{I}_{i+1}$, it is possible to place at least two one-dimensional voxels $\mathbf{V}(k) = \{x \,|\, |x-k| \leq 1\}$ for $k \in \mathbf{Z}$.

There exist 3^n points in the region

$$\mathbf{C}(\boldsymbol{a}) = \{\boldsymbol{x} | |\boldsymbol{x} - \boldsymbol{a}|_\infty \leq 1\} \tag{45}$$

around a point $\boldsymbol{a}$. For the point $\boldsymbol{a} \in \mathbf{Z}^n$, if

$$(\partial_+ \mathbf{F} \cup \partial_- \mathbf{F}) \bigcap \mathbf{C}(a) = \mathbf{C}(a), \tag{46}$$

the point $\boldsymbol{a}$ lies on a locally flat manifold. Furthermore, if the point $\boldsymbol{a}$ lies on a corner, the relation

$$(\partial_+ \mathbf{F} \cup \partial_- \mathbf{F}) \bigcap \mathbf{C}(a) \subset \mathbf{C}(a) \tag{47}$$

is satisfied. Equation (47) implies the relation $|(\partial_+ \mathbf{F} \bigcup \partial_- \mathbf{F}) \bigcap \mathbf{C}(a)| < 3^n$.

The corner points of a digital object $\mathbf{F}$ may separate both the internal boundary $\partial_- \mathbf{F}$ and external boundary $\partial_- \mathbf{F}$ into portions. Using the corners of the internal and external boundaries of the complement $\overline{\mathbf{F}}$ of the digital object, we refine the connectivity of both the internal and external boundaries.

Definition 11. *If there exists at least one path between all pairs of points on the internal and external boundaries, these boundaries are called the refined internal and external boundaries, respectively.*

We call the point sets

$$\mathbf{C}_- = (\partial_- \overline{\mathbf{F}} \bigcup \partial_+ \mathbf{F}) \setminus (\partial_- \overline{\mathbf{F}} \bigcap \partial_+ \mathbf{F}), \tag{48}$$

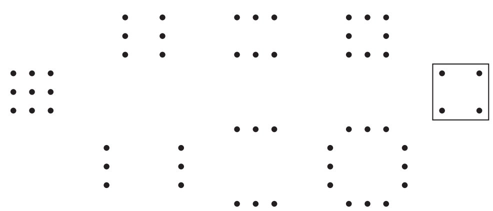
$$\mathbf{C}_+ = (\partial_+ \overline{\mathbf{F}} \bigcup \partial_- \mathbf{F}) \setminus (\partial_+ \overline{\mathbf{F}} \bigcap \partial_- \mathbf{F}) \tag{49}$$

the singular points, which disturb the connectivity along the boundary curves. The refined internal and external boundaries

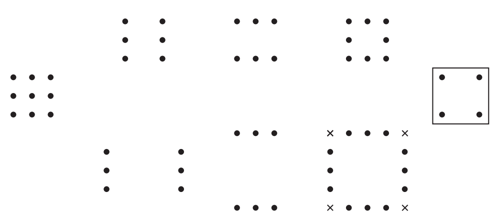
$$\overline{\partial_- \mathbf{F}} = \partial_- \mathbf{F} \bigcup \mathbf{C}_-, \qquad \overline{\partial_+ \mathbf{F}} = \partial_+ \mathbf{F} \bigcup \mathbf{C}_+. \tag{50}$$

prevent the continuity. The reltions in (50) are expressed as

$$\overline{\partial_- \mathbf{F}} = \partial_- \mathbf{F} \bigcup \partial_+ \overline{\mathbf{F}}, \qquad \overline{\partial_+ \mathbf{F}} = \partial_+ \mathbf{F} \bigcup \partial_- \overline{\mathbf{F}}. \tag{51}$$



(a)



(b)

Fig. 5. Refinement operation and boundary detection. (a) Union of the internal and external boundaries. (b) Refinement operations at the corners preserve the continuity of the internal and external boundaries.

Figure 5 illustrates the refinement operation for boundary detection. Refinement operations at the corners preserve the continuity of the internal and external boundary.

$\partial_{\pm}\overline{\mathbf{F}}$ is numerically computed by

$$\partial_{\pm}\overline{\mathbf{F}} = \{\partial_{\pm}(\mathbf{H} \setminus \mathbf{F})\} \setminus \partial_{\pm}\boldsymbol{F} \tag{52}$$

for a large hypercube $\mathbf{H}$, which encloses $\mathbf{F}$ with the condition

$$\min_{\boldsymbol{x}\in(\mathbf{H}\setminus\mathbf{F}),\boldsymbol{y}\in\mathbf{F}} |\boldsymbol{x} - \boldsymbol{y}| \geq 3 \tag{53}$$

on the isothetic lines $\boldsymbol{z} = \boldsymbol{a} + t\boldsymbol{e}_i$ for $\boldsymbol{a} \in \mathbf{Z}^n$.

We define the digital set gradient on the boundary as

$$\partial\mathbf{F} = \left(\bigcup_{\boldsymbol{x}\in\overline{\partial_+\mathbf{F}}} \mathbf{V}(\boldsymbol{x})\right) \bigcap \left(\bigcup_{\boldsymbol{x}\in\overline{\partial_-\mathbf{F}}} \mathbf{V}(\boldsymbol{x})\right). \tag{54}$$

$\partial\mathbf{F}$ is the boundary of the embedding $\mathcal{F}$ of the object $\mathbf{F}$, that is, $\partial\mathbf{F} = \partial\mathcal{F}$. Then, $\partial\mathbf{F}$ is an isothetic Nef-polytope whose vertices and faces lie on the dual grid $\mathbf{D}^n$. Therefore, we have the next theorem.

Theorem 2. *$\partial\mathcal{F}$ is a union of $(n-1)$ simplices [6] in the dual grid.*

For a thin object $\mathbf{T}$ in $\mathbf{Z}^n$, we call the embedding of $\mathbf{T}$ in $\mathbf{R}^n$

$$\mathcal{T} = \bigcup_{\boldsymbol{x}\in\mathbf{T}} \mathbf{V}(\boldsymbol{x}) \tag{55}$$

an imperfect voxel object.

Definition 12. *In $\mathbf{R}^n$, if the complement of voxel object $\mathcal{P}$ is an imperfect voxel object, we call $\mathcal{P}$ a perfect voxel object.*

In a perfect voxel object, which is the Euclidean embedding of a thick object in $\mathbf{Z}^n$, any imperfect voxel object is contained as connected components, although imperfect voxel objects are permissible for embeddings of point sets based on the well-composed sets [7].

Let $[\partial\mathbf{F}] = [\partial\mathcal{F}]$ be the closure of $\partial\mathbf{F} = \partial\mathcal{F}$. Since a voxel is a simplex in $\mathbf{D}^n$, we have the next theorem.

Theorem 3. *The closure of $[\partial\mathbf{F}] = [\partial\mathcal{F}]$ is an n-complex in the dual grid.*

For the thickness of $[\partial\mathbf{F}] = [\partial\mathcal{F}]$, we have the next theorem.

Theorem 4. *The thickness of the complement of $[\partial\mathbf{F}] = [\partial\mathcal{F}]$ is at least two voxels.*

(Proof) On any isothetic digital line

$$\mathbf{L}(k, \boldsymbol{z}) = \lambda \boldsymbol{e}_k + \boldsymbol{z}, \ \ \boldsymbol{z} \in \mathbf{Z}^n$$

parallel to the vector $\boldsymbol{e}_k$, the linear object

$$\mathbf{F}(k, \boldsymbol{z}) = \mathbf{F} \bigcap \mathbf{L}(k, \boldsymbol{z})$$

is a thick one-dimensional object. The thickness of the complement of the embedding $\mathcal{F}(k, \boldsymbol{z})$ is at least two voxels. □

Theorem 4 implies the following statement on the embedding of digital objects in a digital space into Euclidean space.

Theorem 5. *An isothetic Nef-polytope $\mathcal{F}$ and its complement are perfect voxel objects.*

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References

1. Imiya, A., Eckhardt, U., The Euler characteristics of discrete objects and discrete quasi-objects, CVIU, **75** 307-318 1999.
2. Imiya, A., Geometry of three-dimensional neighbourhood and its applications (in Japanese), Trans. of Information Processing Society of Japan, **34**, 2153-2164, 1993.
3. Klette, R., Rosenfeld, A.: *Digital Geometry: Geometric Methods for Digital Picture Analysis,* Morgan Kaufmann, 2004.
4. Serra, J., *Image Analysis and Mathematical Morphology*, Academic Press, 1982.
5. Bieri, H., Nef, W., A sweep-plane algorithm for computing the volume of polyhedra represented in Boolean form, Linear Algebra and Its Applications, **52/53**, 69-97, 1983.
6. Herman, G.T., *Geometry of Digital Spaces* , Birkhaeuser, 1998.
7. Latecki L. J. Well-composed sets, *Advances in Imaging and Electron Physics* **112**, 95-163, 2000.